Fluid dynamics

Lecture at the university of Trondheim
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Contents

Literature

1. L.D. Landau and E.M. Lifshitz, Fluid Mechanics, Pergamon Press, New York.

2. G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge.

3. H. Lamb, Hydrodynamics, Dover Publications, New York.

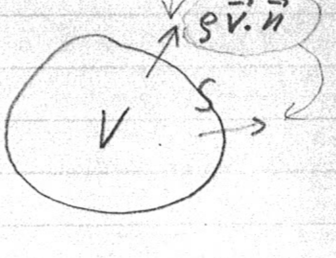
1 Ideal Fluids

1.1 The equation of continuity

Fluid dynamics concerns itself with the motion of fluids (liquids and gases). The description of the system is macroscopic; the fluid is regarded as a continuous medium. If we talk about a "small" <u>fluid element</u> it is still assumed to be large enough to contain a very large number of molecules. Small therefore means small compared to the size of e.g. the container but large compared to the distance between the molecules.

The description of the state of a moving fluid is effected by means of functions which give the velocity of the fluid $\vec{v}(\vec{r},t)$, in a point $\vec{r}$ in space at a time t and any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(\vec{r},t)$ and the density $\varrho(\vec{r},t)$. As is well known all thermodynamic quantities are determined by the value of any two of them.
The description of the fluid is thus given in terms of five fields: three velocity fields and two thermodynamic fields as e.g. p and ϱ.

In order to derive the equation of continuity we use mass conservation. This conservation law implies that the mass in a volume V may only change by flow $\varrho\vec{v}\cdot\vec{n}$ through the surface S. This gives



$$\frac{d}{dt}\int_V \varrho(\vec{r},t)\,d\vec{r} = -\int_S \varrho(\vec{r},t)\,\vec{v}(\vec{r},t)\cdot\vec{n}(\vec{r})\,dS \qquad (1.1.1)$$

where $\vec{n}(\vec{r})$ is the outward normal on the surface of V. Note the fact that the volume is chosen independent of the time. Using Green's identity the above equation becomes

$$\int_V \left[\frac{\partial}{\partial t}\varrho(\vec{r},t) + \operatorname{div}\left(\varrho(\vec{r},t)\,\vec{v}(\vec{r},t)\right)\right] d\vec{r} = 0 \qquad (1.1.2)$$

In view of the fact that this is true for all possible choices of V it follows that the integrand is zero.

$$\frac{\partial}{\partial t}\varrho(\vec{r},t) + \mathrm{div}\left(\varrho(\vec{r},t)\,\vec{v}(\vec{r},t)\right) = 0 \qquad (1.1.3)$$

<u>equation of continuity</u>

This is simply the differential form of the equation for mass conservation which was given in integral form in eq. (1.1.1). The vector $\varrho\vec{v}$ is called the mass flux density.

If one considers the change of a certain quantity θ in a fluid element (θ may be e.g. the density ϱ or the energy density etc.) there are two contributions which one may distinguish. $\partial\theta/\partial t$ is the <u>local rate of change</u> due to temporal changes at a fixed position. To find the total rate of change in a fluid element one must add the <u>convective rate of change</u> $\vec{v}.\mathrm{grad}\,\theta$ due to transport of the element to a different position. It is convenient to define, $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}.\vec{\nabla} \qquad (1.1.4)$$

$D\theta/Dt$ now gives the total rate of change of θ in a fluid element.

Using the above definition the equation of continuity may be written in the following form

$$\frac{1}{\varrho}\frac{D\varrho}{Dt} = -\vec{\nabla}.\vec{v} \qquad (1.1.5)$$

$\vec{\nabla}.\vec{v}$ is called the <u>rate of expansion</u> of a fluid element. A fluid is said to be incompressible if the density in a fluid element does not change as a function of the time

$$\frac{D\varrho}{Dt} = 0 \quad \Rightarrow \quad \vec{\nabla}.\vec{v} = 0 \qquad (1.1.6)$$

Such velocity fields are called <u>solinoidal</u>. In many practical situations also for gases the flow is such that the density in the fluid elements does not change and the fluid may be said to be incompressible.

1.2 Analysis of relative motion near a point

The force exerted by one fluid element on an adjacent element depends on the way the fluid is being deformed by its motion. It is therefore usefull, as a preliminary to dynamical considerations, to analyse the character of the motion in the neighbourhood of any point. Expanding the velocity field around $\vec{r}_0$ we have

$$\delta\vec{v}(\vec{r}) \equiv \vec{v}(\vec{r}) - \vec{v}(\vec{r}_0) = \delta\vec{r}\cdot\left[\vec{\nabla}\,\vec{v}(\vec{r})\right]_{\vec{r}=\vec{r}_0} \qquad (1.2.$$

The gradient of the velocity field, $\vec{\nabla}\vec{v}$, is a second order tensor field which may be decomposed in a symmetric and an antisymmetric part

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) \text{ and } \xi_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial r_j} - \frac{\partial v_j}{\partial r_i}\right) \qquad (1.2.$$

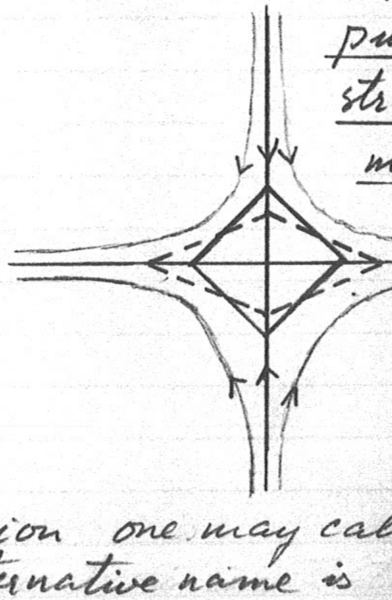
Using these tensors we may write

$$\delta\vec{v} = \delta\vec{v}^{(s)} + \delta\vec{v}^{(a)} = \vec{\vec{e}}\cdot\vec{r} + \vec{\vec{\xi}}\cdot\vec{r} \qquad (1.2.$$

The trace of $\vec{\vec{e}}$ is equal to the rate of expansion

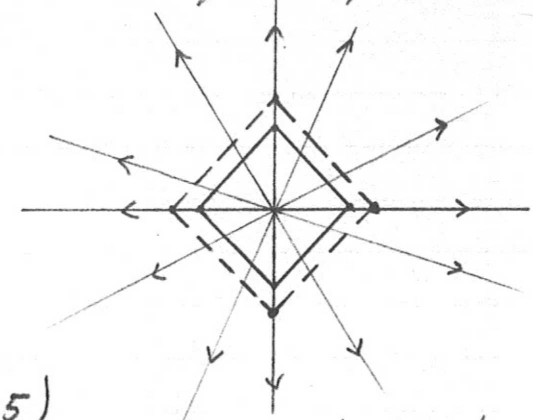
$$Tr\,\vec{\vec{e}} = \vec{\nabla}\cdot\vec{v} \qquad (1.2.$$

Because of the fact that $\vec{\vec{e}}$ is symmetric one may choo[se] three orthogonal axis such that $\vec{\vec{e}}$ becomes diagonal if the x, y and z coordinates are along these axis. If $Tr\,\vec{\vec{e}}=0$ as in the incompressible case the fluid flow is the one given, for a two dimensional case, in the figure. In general one calls δv^s <u>straining motion</u> and in the special case that $Tr\,\vec{\vec{e}}=0$ so that there is no expansion one may call it <u>pure straining motion</u>. An alternative name is <u>elongational motion</u>.



<u>pure straining motion.</u>

isotropic expansion



In general one may always decompose straining motion in pure straining motion and <u>isotropic expansion</u>. For this purpose one writes

$$\vec{\vec{e}} = \left[\vec{\vec{e}} - \tfrac{1}{3}\left(\mathrm{Tr}\,\vec{\vec{e}}\right)\vec{\vec{I}} \right] + \tfrac{1}{3}\left(\mathrm{Tr}\,\vec{\vec{e}}\right)\vec{\vec{I}} \qquad (1.2.5)$$

The term between square brackets corresponds to the pure straining part and the other to isotropic expansion. Note that $\vec{\vec{I}}$ is just the unit tensor which has as trace 3.

Turning to $\delta\vec{v}^{(a)}$ we see that ξ is anti-symmetric with only three independent components. One may write ξ in terms of

$$\vec{\omega} \equiv \mathrm{rot}\,\vec{v} \qquad \text{or} \qquad \omega_i \equiv \epsilon_{ijk} \nabla_j v_k \qquad (1.2.6)$$

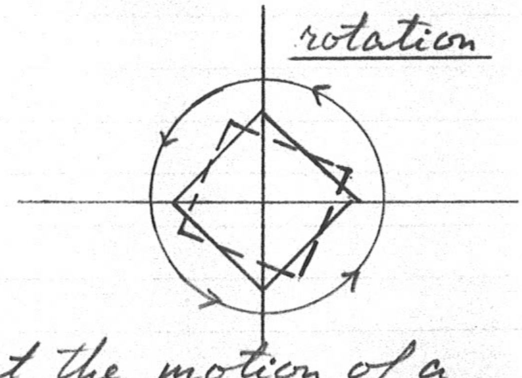
, where ϵ_{ijk} is the Levi-Civita tensor and a summation convention over double indices has been used, as

$$\xi_{ij} = -\tfrac{1}{2}\epsilon_{ijk}\,\omega_k \qquad (1.2.7)$$

This gives for $\delta\vec{v}^{(a)}$

$$\delta v_i^{(a)} = \xi_{ij}\,\delta r_j = -\tfrac{1}{2}\epsilon_{ijk}\,\delta r_j\,\omega_k \Longrightarrow \delta\vec{v}^{(a)} = \tfrac{1}{2}\vec{\omega}\wedge\delta\vec{r} \qquad (1.2.8)$$

rotation

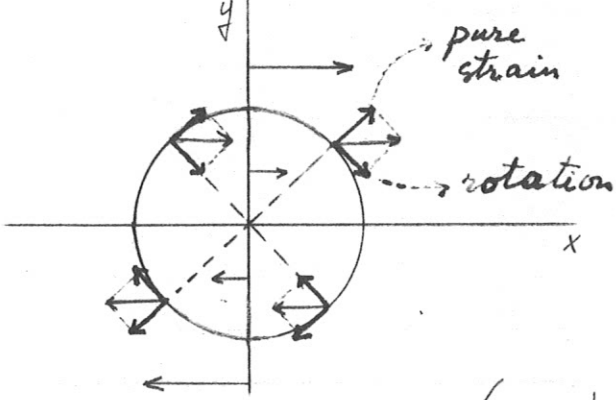


The motion corresponds to simple <u>rotation</u> of the fluid element. In this case the fluid element is not deformed during its motion it rotates so to say as a solid piece of matter.

In summery we have seen that the motion of a fluid element is the sum of a uniform translation with velocity $\vec{v}(\vec{r}_0)$, pure straining, isotropic expansion, rigid body rotation with angular velocity $\frac{1}{2}\vec{\omega}$.

A type of relative motion which occurs often in practice is a <u>simple shearing motion</u>, in which plane layers of fluid slide over one another. The relative velocity $\delta\vec{u}$ has the same direction everywhere in this case. With an appropriate choice of axis one has in fact

$$\delta\vec{v} = (ay, 0, 0) \qquad (1.2.9)$$



In this case

$$\vec{\vec{e}} = \tfrac{1}{2}a\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1.2.10)$$

$$\vec{\vec{\xi}} = \tfrac{1}{2}a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff \vec{w} = -a(0,0,1) \qquad (1.2.11)$$

The principle rates of strain are given by the eigenvalues of $\vec{\vec{e}}$. One finds $\frac{1}{2}a$ in the $\frac{1}{\sqrt{2}}(1,1,0)$ direction, $-\frac{1}{2}a$ in the $\frac{1}{\sqrt{2}}(1,-1,0)$ direction and zero in the $(0,0,1)$ direction.

Simple shearing motion is thus a superposition of pure straining motion and rigid rotation. There is no expansion involved.

1.3 Euler's equation for an ideal fluid

In order to obtain an equation of motion for the velocity field one uses Newton's law for a fluid element. The force acting on the fluid element is due to two sources. One is the variation of the hydrostatic pressure p as a function of position. On one side of the fluid element p is slightly smaller than on the other side and the difference gives a force on the fluid element. This gives a force density (per unit of volume) $-\operatorname{grad} p$. The other possible force $\vec{F}$ is due to external sources as for instance a gravitational force or an electric force if the fluid particles are charged. $\vec{F}$ is given per unit of mass rather than volume. The resulting equation for the velocity field is

$$\varrho \frac{D\vec{v}}{Dt} = -\operatorname{grad} p + \varrho \vec{F} \qquad (1.3.1)$$

Euler's equation

This equation is one of the fundamental equations of fluid mechanics

Using the continuity equation one may also write Euler's equation in the following form

$$\frac{\partial}{\partial t} \varrho \vec{v} = -\operatorname{div}(\varrho \vec{v}\vec{v}) - \operatorname{grad} p + \varrho \vec{F} \qquad (1.3.2)$$

This equation is most easily written in integral form.

If the fluid is at rest the Euler equation reduces to

$$\operatorname{grad} p = \varrho \vec{F} \qquad (1.3.3)$$

If the fluid is also in thermal equilibrium the temperature is the same in every point. Using the following thermodynamic relation

$$d\Phi = -s\,dT + \frac{1}{\varrho}\,dp \qquad (1.3.4)$$

, where Φ is the Gibbs Function and s the entropy density, one may write eq. (1.3.3) in the form

$$\operatorname{grad} \Phi = \vec{F} \qquad (1.3.5)$$

For a conservative force field one may write

$$\vec{F} = -\operatorname{grad} \varphi \qquad (1.3.6)$$

where φ is a scalar potential. It then follows from eq. (1.3.5) that

$$\Phi + \varphi = \text{constant} \qquad (1.3.7)$$

throughout the system. This condition is the usual condition for equilibrium in an external field which may also be found using statistical physics.

As we shall discuss in the second section Euler's equation is not completely correct. If one uses this equation one neglects frictional forces which occur for straining motion. In the present case the fluid layers in e.g. shearing motion slide frictionless over one another. For many purposes the Euler equation is enough.

Neglecting frictional forces implies that the kinetic energy is not converted into heat. Such a process would lead to an increase of the total entropy in the system. If one also neglects heat conduction the total entropy in the system becomes a conserved quantity. The entropy density only changes due to convection. Using s as the entropy density per unit of mass one then has

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s = 0 \qquad (1.3.8)$$

In other words the entropy content of a fluid element, which flows along with the fluid, is constant. The motion of the fluid is <u>adiabatic</u> and a fluid in which the motion is adiabatic is called <u>ideal</u>.

It should be realized that the adiabatic equation usually takes a much simpler form. In most cases s is constant throughout the volume at some initial time. In that case it follows from eq. (1.3.8) that s is constant at all times

$$s(\vec{r},t) = \text{constant} \qquad (1.3.9)$$

Such motion is called <u>isentropic.</u>

The equations of motion have to be supplemented with boundary conditions. At a solid surface such a condition is simply that the fluid cannot penetrate in the solid and consequently the component of the velocity normal to the surface is zero. This is sufficient for ideal fluids where there is no friction between the wall and fluid flowing along the wall.

The equation of continuity, Euler's equation and the boundary condition completely determine the fluid motion in an ideal fluid for isentropic motion.

Using the following thermodynamic relation

$$dh = T ds + \frac{1}{\rho} dp \tag{1.3.10}$$

, where h is the enthalpie density per unit of mass, the Euler equation becomes

$$\frac{D\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}.\vec{\nabla}\vec{v} = -\vec{\nabla} h + \vec{F} \tag{1.3.11}$$

Using the following formula well known in vector analys

$$\frac{1}{2}\vec{\nabla}|\vec{v}|^2 = \vec{v} \wedge (\mathrm{rot}\,\vec{v}) + \vec{v}.\vec{\nabla}\vec{v} \tag{1.3.12}$$

one obtains

$$\frac{\partial}{\partial t}\vec{v} - \vec{v}\wedge(\mathrm{rot}\,\vec{v}) = -\vec{\nabla}\left(h + \frac{1}{2}|\vec{v}|^2\right) + \vec{F} \tag{1.3.13}$$

For the rotation of velocity field $\vec{w} = \mathrm{rot}\,\vec{v}$ this gives

$$\frac{\partial}{\partial t}\vec{w} - \mathrm{rot}\,(\vec{v}\wedge\vec{w}) = \mathrm{rot}\,\vec{F} \tag{1.3.14}$$

If the external force field is conservative, eq. (1.3.6), the last two equations become

$$\frac{\partial}{\partial t}\vec{v} - \vec{v}\wedge\vec{w} = -\vec{\nabla}\left(h + \varphi + \frac{1}{2}|\vec{v}|^2\right) \tag{1.3.15}$$

and

$$\frac{\partial}{\partial t}\vec{w} - \mathrm{rot}\,(\vec{v}\wedge\vec{w}) = 0 \tag{1.3.16}$$

Both equations lead to rather usefull and wellknown theorems for isentropic flow which we will discuss in the following paragraphs.

1.4 Bernoulli's equation for steady flow

The equations of fluid dynamics are much simplified in the case of steady flow, i.e. $\partial\vec{v}/\partial t = 0$. Eq. (1.3.15) for isentropic flow then reduces to

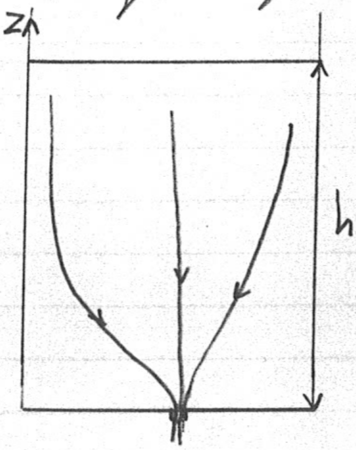
$$\vec{v} \wedge \vec{w} = \nabla\left(h + \varphi + \tfrac{1}{2}|\vec{v}|^2\right) \qquad (1.4.1)$$

We now introduce the concept of a streamline. Streamlines are lines such that the tangents to the streamlines give the direction of the velocity field in every point. Similarly we define vortex lines as lines such that the tangents to the vortex lines give the direction of $\vec{w}$. It then follows from eq. (1.4.1) that

$$h + \varphi + \tfrac{1}{2}|\vec{v}|^2 = \text{constant along stream lines as well as vortex lines}$$

Bernoulli's equation (1.4.2)

This equation may often be used in a very simple way to calculate velocities



Consider e.g. water in a big container with a small hole in the bottom. The question is to find the the velocity of the fluid which spouts out of the hole in terms of the height of the water in the container h and the gravitational potential $\varphi = gz$

As the hole is small h may be assumed constant. The pressure of the air is the same above the water as well as in the hole. The density of water is also the same in both places. Thus h is also the same in both places. φ changes an amount hg from top to bottom. $\vec{v}$ is practically zero near the top and we therefore find

$$|\vec{v}| = \sqrt{2gh} \qquad (1.4.3)$$

as velocity of the flow in the hole.

1.5 Conservation of circulation

The integral

$$\Gamma \equiv \oint \vec{v} \cdot d\vec{\ell} = \int_S (\mathrm{rot}\,\vec{v}) \cdot d\vec{S} = \int_S \vec{\omega} \cdot d\vec{S} \qquad (1.5.1)$$

is called the velocity circulation round that contour. S is an arbitrary surface with the contour as circumference. We used a theorem due to Stokes in order to write the integral along the contour as a surface integral. The contour is chosen to be a "fluid contour" in the sense that it flows along with the fluid. The time derivative of the velocity circulation is

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint \vec{v} \cdot d\vec{\ell} = \oint \frac{D\vec{v}}{Dt} \cdot d\vec{\ell} + \oint \vec{v} \cdot \frac{D}{Dt} d\vec{\ell} \qquad (1.5.2)$$

Since rate of change in the location of the contour is given by the velocity we have

$$\vec{v} \cdot \frac{D}{Dt} d\vec{\ell} = \vec{v} \cdot d\vec{v} = \frac{1}{2} d\,|\vec{v}|^2$$

This implies that the second contour integral in eq. (1.5.2) is zero. In the first we substitute eq. (1.3.11) for a conservative force field, eq. (1.3.6),

$$\frac{D\vec{v}}{Dt} = -\vec{\nabla}(h+\varphi) \qquad (1.5.3)$$

We then find, using again Stokes,

$$\frac{d\Gamma}{dt} = -\oint \vec{\nabla}(h+\varphi) \cdot d\vec{\ell} = -\int_S \mathrm{rot}\left(\vec{\nabla}(h+\varphi)\right) \cdot d\vec{S}$$

and because of the fact that the rotation of a gradient is zero one finds

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint \vec{v} \cdot d\vec{\ell} = 0 \qquad (1.5.4)$$

Kelvin's theorem or law of conservation of circulation.

It should be emphasised that this theorem is only true for isentropic flow.

1.6 Incompressible fluids

In a great many cases of the flow of liquids (and also of gases), their density may be supposed invariable, i.e. constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of fluid elements and we speak of <u>incompressible flow</u>.

The general equations are much simplified in this case. The equation of continuity becomes

$$\operatorname{div}\vec{v} = 0 \tag{1.6.1}$$

Euler's equation may be written as

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}\left(\frac{p}{\rho}\right) + \vec{F} \tag{1.6.2}$$

As in every fluid element the density as well as the entropy are given in an ideal incompressible fluid, all <u>thermodynamic state variables</u> are known in these fluid elements. It should be realized that the pressure p is not a thermodynamic state variable in an incompressible fluid. It is a <u>purely mechanical response</u> of the fluid element such as to assure that it is not compressed by the forces acting on it. In fact $p(\vec{r},t)$ should be chosen such that $\operatorname{div}\vec{v}=0$. An equation for p is found if one takes the divergence of eq. (1.6.2) and uses $\operatorname{div}\vec{v}=0$

$$\Delta\left(\frac{p}{\rho}\right) = \vec{\nabla}\cdot\vec{F} - \frac{\partial}{\partial t}\operatorname{div}\vec{v} - \operatorname{div}(\vec{v}\cdot\vec{\nabla}\vec{v})$$

$$= \vec{\nabla}\cdot\vec{F} - (\vec{\nabla}\vec{v}):(\vec{\nabla}\vec{v}) \tag{1.6.3}$$

Here $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. Furthermore the double dot is defined by

$$(\vec{\nabla}\vec{v}):(\vec{\nabla}\vec{v}) = (\nabla_i v_j)(\nabla_j v_i) \tag{1.6.4}$$

and may be used between any pair of two index tensors.

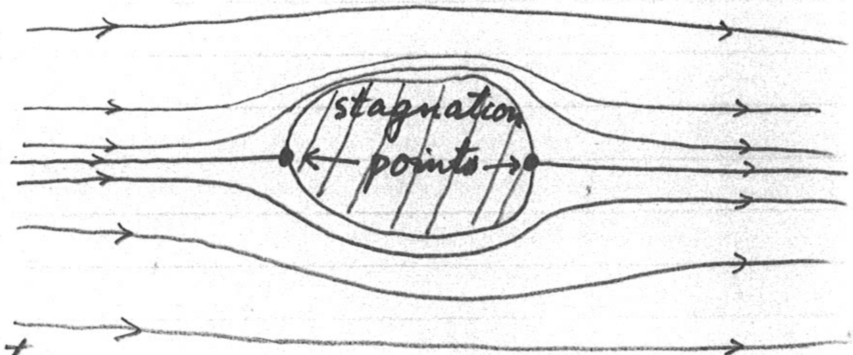
The pressure distribution in an incompressible fluid is thus caused by on the one hand the divergence of the external force and on the other hand by the gradient of the velocity field; this last contribution is due to the convective term in Euler's equation.

If we compare eq. (1.6.2) with eq. (1.3.11) we see that in incompressible ideal fluids p/ρ replaces the enthalpie h used for isentropic ideal fluids. As a consequence we now find

$$\frac{p}{\rho} + \varphi + \tfrac{1}{2}|\vec{v}|^2 = \text{constant along streamlines as well as vortex lines}$$

<u>Bernoulli's equation</u> (1.6.5)

instead of the relation containing h given in eq(1.4.2). This equation is again very convenient for certain purposes. Consider e.g. the case that there is no external force, $\varphi = 0$. It then follows from eq. (1.6.5) that the pressure will be largest in <u>stagnation points</u>, i.e. points where $\vec{v} = 0$. Consider e.g. flow past a solid body which is kept fixed in the fluid. The velocity is $\vec{v}_0$ and the pressure p_0 far away from the solid body. There are two stagnation points on the surface. In these stagnation points the pressure has its maximum value given by



$$p\,(\text{stagnation points}) = p_0 + \tfrac{1}{2}|\vec{v}_0|^2 \qquad (1.6.6)$$

Notice the important fact that the pressure is not only higher in front of the body where the fluid is so to say incident on the surface but also behind the body where it is pulled away. If the body is symmetric one may in fact show that there is no net force on the body which one may intuitively have assumed to be the case.

1.7 Energy density and flux

It is interesting to consider the density and the flux of energy in a fluid. The energy density in a fluid element is the sum of the internal energy u, the external potential energy (the force field is conservative) and the kinetic energy

$$\rho\left(u + \varphi + \tfrac{1}{2}|\vec{v}|^2\right) = \text{total energy density} \tag{1.7.1}$$

Using thermodynamic identities, the equation of continuity, Euler's equation and entropy conservation, eq. (1.3.8), one may show that

$$\frac{\partial}{\partial t}\left[\rho\left(u + \varphi + \tfrac{1}{2}|\vec{v}|^2\right)\right] = -\,\mathrm{div}\left[\rho\vec{v}\left(h + \varphi + \tfrac{1}{2}|\vec{v}|^2\right)\right] \tag{1.7.2}$$

, cf. Landau-Lifshitz page 10-12. We thus find

$$\rho\vec{v}\left(h + \varphi + \tfrac{1}{2}|\vec{v}|^2\right) = \text{energy flux density} \tag{1.7.3}$$

The fact that the enthalpie appears in this case rather than the internal energy has a simple physical significance. The difference between $\rho u\vec{v}$ and $\rho h\vec{v}$ is equal to $p\vec{v}$ which is the work done by pressure forces during the flow. Using Bernoulli's theorem it follows that the energy current changes only because $\rho\vec{v}$ changes along stream or vortex lines, this both in direction as well as in size.

In the incompressible fluid one must again replace the enthalpie h by P/ρ and one finds

$$\rho\vec{v}\left(\frac{P}{\rho} + \varphi + \tfrac{1}{2}|\vec{v}|^2\right) = \text{energy flux density in an incompressible fluid} \tag{1.7.4}$$

As both ρ and s are constant in this case u is also constant in every fluid element; or everywhere in the isentropic case. u is no longer a relevant quantity in the flow problem and the use of

$$\rho\left(\varphi + \tfrac{1}{2}|\vec{v}|^2\right) = \text{total energy in an incompressible fluid} \tag{1.7.5}$$

is more appropriate.

1.8 Potential flow in incompressible fluids

The equations for the flow in an incompressible fluid are particularly simple if

$$\mathrm{rot}\,\vec{v} = \vec{w} = 0 \qquad (1.8.1)$$

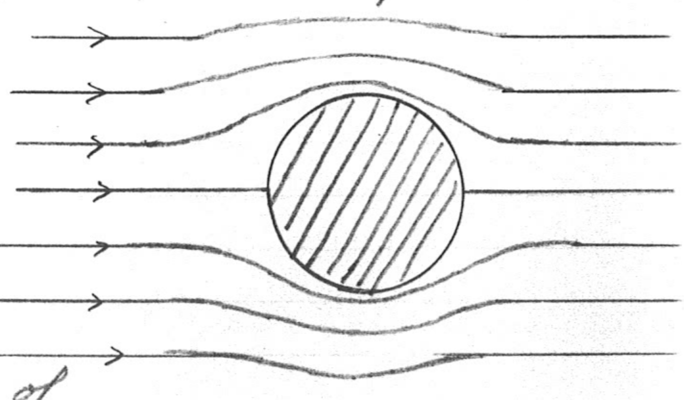
This is a trivial solution of eq. (1.3.14) if $\mathrm{rot}\,\vec{F} = 0$ which is the case for conservative force fields. It follows from eq. (1.8.1) that we may write $\vec{v}$ as the gradient of a potential

$$\vec{v} = \mathrm{grad}\,\phi \qquad (1.8.2)$$

In view of the incompressibility ϕ satisfies

$$\mathrm{div}\,\vec{v} = \mathrm{div}\,\mathrm{grad}\,\phi = \boxed{\Delta\phi = 0} \quad \underline{\text{Laplace equation}} \qquad (1.8.3)$$

For some flow problems this solution method is very convenient. Consider e.g. flow around a sphere. The center of the sphere is located at $\vec{r} = 0$. The radius is a. The velocity of the fluid far from the sphere is $\vec{u}$. The sphere is at rest and therefore the normal component of $\vec{v}$ is zero on the surface of the sphere. We may easily construct a solution using elementary solutions of the Laplace equation:



$$\frac{1}{r} \;;\; \vec{q}\cdot\vec{\nabla}\frac{1}{r} \;;\; \vec{\vec{Q}} : \vec{\nabla}\vec{\nabla}\frac{1}{r} \quad \text{etc.} \qquad (1.8.4)$$

where $\vec{q}$ is a constant vector and $\vec{\vec{Q}}$ a constant tensor. The solution $1/r$ is well known and the others follow from this solution by differentiation. The simplest choice of ϕ to solve the problem is

$$\phi = \vec{u}\cdot\vec{r} + c\,\vec{u}\cdot\vec{\nabla}\frac{1}{r} = \vec{u}\cdot\vec{r}\left(1 - \frac{c}{r^3}\right) \qquad (1.8.5)$$

where the constant c has to be chosen such that the boundary condition is satisfied. The resulting velocity field is

$$\vec{v} = \vec{\nabla}\phi = \vec{\nabla}\left[\vec{u}\cdot\vec{r}\left(1-\frac{c}{r^3}\right)\right] = \left(\vec{\nabla}\,\vec{u}\cdot\vec{r}\right)\left(1-\frac{c}{r^3}\right) + \vec{u}\cdot\vec{r}\;\vec{\nabla}\left(1-\frac{c}{r^3}\right)$$

$$= \vec{u} - \frac{c}{r^3}\,\vec{u}\cdot\left(1 - 3\,\frac{\vec{r}\vec{r}}{r^2}\right) \qquad (1.8.6$$

On the surface of the sphere we find as normal compon

$$v_n = \vec{n}\cdot\vec{v}\Big|_{r=a} = \vec{n}\cdot\vec{u}\left(1 + 2\,\frac{c}{a^3}\right) = 0 \qquad (1.8.7$$

It thus follows that

$$c = -\tfrac{1}{2}a^3 \implies \vec{v} = \vec{u} + \tfrac{1}{2}\left(\frac{a}{r}\right)^3\vec{u}\cdot\left(1 - 3\,\frac{\vec{r}\vec{r}}{r^2}\right) \qquad (1.8.8$$

This is the solution. $\vec{u}$ is so to say the incident velocity field and the contribution proportion to r^{-3} is the result of the interaction with the sphere.

In order to find the pressure in terms of ϕ we write the Euler equation in the form given in eq. (1.3.1 but now for the incompressible fluid

$$\frac{\partial}{\partial t}\vec{v} - \vec{v}\wedge\vec{w} = -\vec{\nabla}\left(\frac{p}{\rho} + \varphi + \tfrac{1}{2}|\vec{v}|^2\right) \qquad (1.8.$$

In the present case this reduces to

$$\frac{\partial}{\partial t}\phi + \frac{p}{\rho} + \varphi + \tfrac{1}{2}|\vec{v}|^2 = 0 \qquad (1.8.$$

where any additive constant has been absorbed in ϕ. This expression may be used to calculate the pressure on the surface of the sphere. The resultin pressure is the same on opposite sides of the sphe and as a consequence the total force on the sphe is indeed zero.

1.9 Two dimensional steady potential flow

If the velocity field depends on only two coordinates, say x and y, and the velocity is everywhere parallel to the x-y plane the flow is said to be two dimensional. From the equation

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y = 0 \qquad (1.9.1)$$

for incompressible twodimensional flow it follows that there exists a socalled stream function ψ in terms of which the velocity may be written as

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x} \qquad (1.9.2)$$

Eq. (1.9.1) is now trivially satisfied. On the other hand one may also write $\vec{v}$ in terms of the potential ϕ in potential flow and we thus have

$$v_x = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad (1.9.3)$$

These relations between the derivatives of ψ and ϕ are the same, mathematically, as the well-known Cauchy-Riemann conditions for the analytic function

$$f(z) \equiv \phi + i\psi \quad \text{with } z \equiv x + iy \qquad (1.9.4)$$

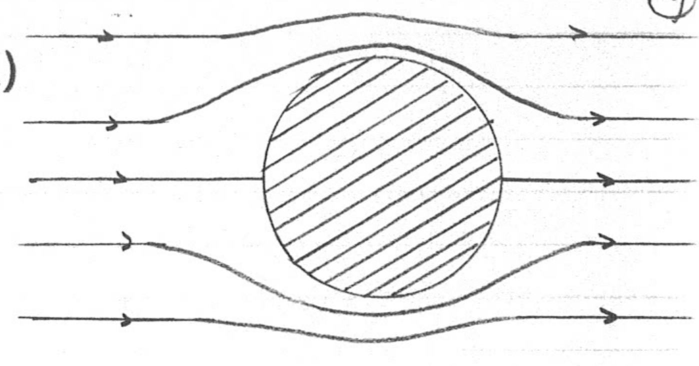
f is the complex potential and df/dz the complex velocity for this problem

$$\frac{df}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = v_x - i v_y \qquad (1.9.5)$$

The extremely usefull fact follows that every analytic function gives a solution to a two dimensional steady potential flow problem in an incompressible fluid. Usually one has the problem and one must search for the corresponding analytic function which is not necessarily allways trivial. This method has nevertheless been extremely usefull.

Example 1

Flow past a cylinder (radius a)
Velocity far from cylinder equal to $\vec{u} = (u,0,0)$
The axis of the cylinder is along the z axis
The solution is given by



$$f = u\zeta + a^2 \frac{u}{\zeta} \qquad (1.9.6$$

The complex velocity is

$$\frac{df}{d\zeta} = u - a^2\frac{u}{\zeta^2} = u - \frac{ua^2}{(x+iy)^2} = u - ua^2\frac{(x-iy)^2}{(x^2+y^2)^2} \qquad (1.9.7$$

This gives

$$v_x = u - ua^2\frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{and } v_y = -2ua^2\frac{xy}{(x^2+y^2)^2} \qquad (1.9.8$$

If we define $r^2 \equiv x^2+y^2$ and $\vec{n} \equiv (x,y,0)/r$ we obtain

$$\vec{v} = \vec{u} + \frac{a^2}{r^2}\vec{u}\cdot(1-2\vec{n}\vec{n}) \qquad (1.9.9$$

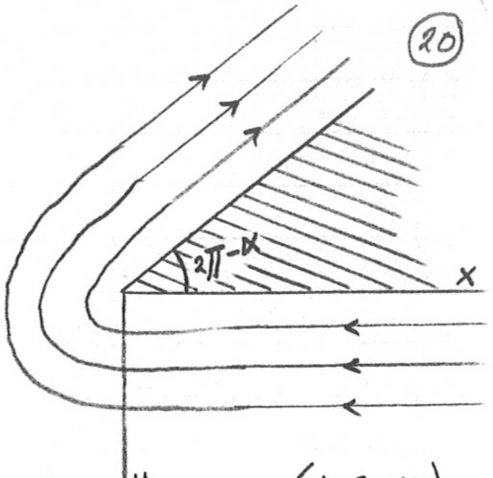
As is easily checked $v_n = \vec{v}\cdot\vec{n} = 0$ on the surface of the cylinder.

One may of course ask how one should in general choose f. In practise there is no real good recipy for this choice. In this case the cylinder is symmetric around the z axis. It is thus not unreasonable to assume that a simple power of ζ should be sufficient as the incident field is also a simple power. It must be an inverse power as the resulting velocity must go to zero for large values of r.

Example 2

Flow past a wedge.

In this case one uses



$$f(z) = \frac{1}{\nu} u \, z^{\nu} \qquad (1.9.10)$$

(1.9.11)

The complex velocity is

$$\frac{df}{dz} = u z^{\nu-1} = u r^{\nu-1} e^{i\theta(\nu-1)} .$$

where $r^2 \equiv x^2+y^2$, $\theta \equiv \operatorname{arctg}\left(\frac{y}{x}\right)$. Using $z^* = r e^{-i\theta}$ we write

$$\frac{df}{dz} = u r^{\nu-2} e^{i\theta\nu} z^* = u r^{\nu-2} (\cos\theta\nu + i\sin\theta\nu)(x-iy) = v_x - i v_y \qquad (1.9.12)$$

The velocity field becomes therefore

$$\vec{v} = u r^{\nu-2} \left[\cos\nu\theta (x,y,0) \quad + \sin\nu\theta (y,-x,0) \right] \qquad (1.9.13)$$

$\downarrow$ this component flows parallel to sides of the wedge

$\downarrow$ this component has a direction orthogonal to the sides of the wedge

We must, in order to make $v_n = 0$, choose

$$\sin\nu\alpha = 0 \quad \Rightarrow \quad \nu = \frac{\pi}{\alpha} \qquad (1.9.14)$$

The velocity thus becomes

$$\vec{v} = u r^{\left(\frac{\pi}{\alpha}-2\right)} \left[\cos\pi\frac{\theta}{\alpha} (x,y,0) \quad + \sin\pi\frac{\theta}{\alpha} (y,-x,0) \right] \qquad (1.9.15)$$

Notice that if
$$\begin{cases} \alpha < \frac{\pi}{2} = 90^\circ & \vec{v} \to 0 \quad \text{if } r \to 0 \\ \alpha > \frac{\pi}{2} = 90^\circ & \vec{v} \to \infty \quad \text{if } r \to 0 \end{cases}$$

2 Viscous fluids

2.1 The Navier - Stokes equation

In the first chapter we considered the adiabatic motion in a socalled ideal fluid. In that case we neglected "frictional forces" which occur in fluids because neighbouring fluid elements do not have the same velocity. In section 1.2 we discussed the various types of relative motion. After substracting the common translational motion we were left with straining motion and simple rotation. In simple rotational motion the fluid rotates as a "solid piece of matter", see section 1.2 for a more precise formulation. Just as with translational motion such rotational motion does not lead to friction. Thus only straining motion leads to frictional forces. Straining motion could be written as the sum of pure straining motion and isotropic expansion. The frictional force may most conveniently be written in terms of the socalled <u>viscous pressure tensor</u>,

$$\rho \vec{F}_{frictional} = - \vec{\nabla} \cdot \vec{\vec{\Pi}} \qquad , \qquad (2.1.1)$$

which simply the flux of momentum of one fluid element to the next due to the friction. The viscous pressure tensor may now be given in terms of the straining tensor $\vec{\vec{e}}$ as

$$\Pi_{ij} = -2\eta\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\,\mathrm{Tr}\,\vec{\vec{e}}\right) - \zeta\,\delta_{ij}\,\mathrm{Tr}\,\vec{\vec{e}}$$

$$= -\eta\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\delta_{ij}\,\mathrm{div}\,\vec{v}\right) - \zeta\,\delta_{ij}\,\mathrm{div}\,\vec{v} \qquad (2.1.2)$$

η is the <u>shear viscosity</u> or more simply the viscosity which is different for different fluid and is a measure of how much the fluid "dislikes" pure straining motion. ζ is the <u>bulk viscosity</u> which is also different for different fluids and is a measure of how much the fluid "dislikes" isotropic expansion. "Disliked" is used here in the sense that the friction will damp the corresponding kind of relative motion and would cause it to disappear if there is no external source.

Adding the frictional force to the Euler equation (1.3.1) one obtains

$$\varrho \frac{D\vec{v}}{Dt} = \varrho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\operatorname{grad}\vec{v}\right) = -\operatorname{grad}p - \operatorname{div}\vec{\vec{\Pi}} + \varrho\vec{F}$$

$$= -\operatorname{grad}p + \eta\Delta\vec{v} + \left(\zeta + \tfrac{1}{3}\eta\right)\operatorname{grad}\operatorname{div}\vec{v} + \varrho\vec{F} \qquad (2.1.3)$$

<u>Navier-Stokes equation</u>

This is the general equation for the velocity of the fluid assuming that η and ζ are constants. In fact the viscosities depend on in particular the temperature but usually not strong enough to take gradients of this temperature dependence into account. One simply uses the viscosities at some constant average temperature. As we shall show in section (2.3) both viscosities are positive,

$$\eta \geqslant 0 \quad \text{and} \quad \zeta \geqslant 0, \qquad (2.1.4)$$

as a consequence of the second law of thermodynamics. In most normal fluids both η and ζ are relatively small. The viscous pressure in such fluids is indeed a linear function of the straining motion and these fluids are called <u>Newtonian fluids</u>. For larger values of the viscosity one must take effects into account like memory and nonlinear dependence on the strain. Such <u>non-Newtonian fluids</u> are extremely interesting but they are not the subject of these lectures.

As we have allready said $\vec{\vec{\Pi}}$ gives the flux of momentum due to friction. Also the hydrostatic pressure contributes to this flux and one may define the total pressure tensor as the combination of these two contributions

$$P_{ij} = p\,\delta_{ij} + \Pi_{ij} \qquad (2.1.5)$$

Note that one may also interprete $(\varrho\vec{v}\vec{v})$, cf. eq. (1.3.2), as a flux of momentum. This is the contribution to this flux due to convection.

In incompressible fluids the Navier-Stokes equation simplifies to

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}.\,\mathrm{grad}\,\vec{v} = -\mathrm{grad}\,\frac{p}{\rho} + \nu \Delta \vec{v} + \vec{F} \qquad (2.1.6)$$

where we defined the kinematic viscosity by

$$\nu \equiv \frac{\eta}{\rho} \qquad (2.1.7)$$

As is clear only one viscosity is important in incompressible flow. Since for many purposes flow is incompressible the shear viscosity η, or equivalently ν, is the important coefficient characterizing the flow. We give below the values of η and ν for various fluids at a temperature of 20°C:

	η (g/cm sec)	ν (cm²/sec)
water	0.010	0.010
air	0.00018	0.150
alcohol	0.018	0.022
glycerine	8.5	6.8
mercury	0.0156	0.0012

These values have been taken from Landau-Lifshitz and are thus rather old. Better values are probably available.

Using the identity (1.3.12) one may write eq. (2.1.6) in the following form, $\vec{w} \equiv \mathrm{rot}\,\vec{v}$,

$$\frac{\partial}{\partial t}\vec{v} - \vec{v} \wedge \vec{w} = -\vec{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\vec{v}|^2\right) + \nu \Delta \vec{v} + \vec{F} \qquad (2.1.6)$$

For steady flow in a conservative force field this becomes

$$\vec{\nabla}\left(\frac{p}{\rho} + \varphi + \frac{1}{2}|\vec{v}|^2\right) = \vec{v} \wedge \vec{w} + \nu \Delta \vec{v} \qquad (2.1.7)$$

Now $\frac{p}{\rho} + \varphi + \frac{1}{2}|\vec{v}|^2$ is no longer constant along the streamlines or vortex lines due to the finite value of the viscosity. If one takes the rotation of eq. (2.1.6) one finds

$$\frac{\partial}{\partial t}\vec{w} - \mathrm{rot}(\vec{v} \wedge \vec{w}) = \nu \Delta \vec{w} + \mathrm{rot}\,\vec{F} \qquad (2.1.8)$$

This equation no longer contains the pressure just as in the case of the ideal fluid. The viscous pressure does modify the time dependence of the vorticity density $\vec{w}$. It damps it in fact. It does not create it if it is not there.

In order to make the description of the velocity field complete one also needs boundary conditions. One usually requires that the fluid velocity is zero on a fixed solid boundary, the so called <u>stick boundary condition</u>. Other boundary conditions are possible but are less common. It should be emphasized that both the normal as well as the tangential velocity vanish , whereas for an ideal fluid the boundary condition requires only the normal velocity to vanish on a fixed solid surface. This essential difference is related to the fact that the Euler equation contains only first order spacial derivatives while the Navier-Stokes equation contains also second derivatives. Why the boundary condition for the tangential velocity can be eliminated in the limit $\eta \to 0$ can be understood in the context of boundary layer theory and is not at all a very trivial matter.

In the general case of a moving solid surface the velocity must be equal to the velocity of the surface.

2.2 Energy density and flux

The presence of the viscous pressure term leads to an additional term $\overleftrightarrow{\Pi} \cdot \vec{v}$ in the energy current. This is simply a work term just like the term $p\vec{v}$ mentioned in section (1.7), where we discussed this same problem for the ideal fluid. In addition to this term one also has the heat flow $\vec{J}$ which is given in terms of the temperature gradient by

$$\vec{J} = -\lambda \, \mathrm{grad}\, T \quad , \qquad (2.2.1)$$

λ is the heat conductivity. Adding these two terms to the energy flow in eq. (1.7.2) one obtains

$$\frac{\partial}{\partial t}\left[\varrho\left(u+\varphi+\tfrac{1}{2}|\vec{v}|^2\right)\right] = -\operatorname{div}\left[\varrho\vec{v}\left(h+\varphi+\tfrac{1}{2}|\vec{v}|^2\right)+\vec{\vec{\Pi}}\cdot\vec{v}+\vec{j}\,\right] \quad (2.2.2$$

This equation is for the total energy. Using $h = u + \frac{p}{\varrho}$ and the equation of continuity one may also write the above equation in the form

$$\varrho\frac{D}{Dt}\left(u+\varphi+\tfrac{1}{2}|\vec{v}|^2\right) = -\operatorname{div}\left(p\vec{v}+\vec{\vec{\Pi}}\cdot\vec{v}+\vec{j}\right) = -\operatorname{div}\left(\vec{\vec{P}}\cdot\vec{v}+\vec{j}\right) \quad (2.2.$$

The irreversible nature of the flow is related to the transfer of kinetic or potential energy into heat. We therefor write, cf. eqs. (2.1.3) and (2.1.5) for a conservative force,

$$\varrho\frac{D}{Dt}\tfrac{1}{2}|\vec{v}|^2 = \varrho\vec{v}\cdot\frac{D}{Dt}\vec{v} = -\vec{v}\cdot\left(\operatorname{div}\vec{\vec{P}}\right) - \varrho\vec{v}\cdot\operatorname{grad}\varphi \quad (2.2.$$

Furthermore we have

$$\varrho\frac{D\varphi}{Dt} = \varrho\vec{v}\cdot\operatorname{grad}\varphi \quad (2.2.$$

Subtracting the last two equations from eq. (2.2.3) one obtains

$$\varrho\frac{Du}{Dt} = -\vec{\vec{P}}:\operatorname{grad}\vec{v} - \operatorname{div}\vec{j} = -p\operatorname{div}\vec{v} - \vec{\vec{\Pi}}:\operatorname{grad}\vec{v} - \operatorname{div}\vec{j} \quad (2.2.($$

where we did use the fact that $\vec{\vec{P}}$ is a symmetric tensor. Here $p\operatorname{div}\vec{v}$ corresponds to a reversible compression term which also appears in an ideal fluid. $\vec{\vec{\Pi}}:\operatorname{grad}\vec{v}$ is the Rayleigh dissipation function and corresponds to an irreversible conversion of kinetic energy into heat.

2.3 The entropy balance equation

We shall now proceed to calculate the increase of the entropy in the system. For this purpose we use the Gibbs relation, $T\,ds = du + p\,d(1/\varrho)$, in a fluid element

$$T\frac{Ds}{Dt} = \frac{Du}{Dt} + p\frac{D\,1/\varrho}{Dt} = \frac{Du}{Dt} - \frac{p}{\varrho^2}\frac{D\varrho}{Dt} \quad (2.3.$$

Using eq. (2.2.6) for the internal energy and the continuity equation (1.1.5) one obtains

$$\varrho \frac{Ds}{Dt} = \frac{\varrho}{T}\frac{Du}{Dt} - \frac{p}{\varrho T}\frac{D\varrho}{Dt} = -\frac{1}{T}\overset{\Rightarrow}{\Pi} : \text{grad}\,\vec{v} - \frac{1}{T}\,\text{div}\,\vec{j}$$

$$= -\,\text{div}\left(\frac{1}{T}\vec{j}\right) - \frac{1}{T}\overset{\Rightarrow}{\Pi} : \text{grad}\,\vec{v} - \frac{1}{T^2}\vec{j}\cdot\text{grad}\,T \qquad (2.3.2)$$

In this equation we may identify the <u>entropy flux</u>

$$\vec{j}_s = \frac{1}{T}\vec{j} = -\frac{\lambda}{T}\,\text{grad}\,T \qquad (2.3.3)$$

and the <u>entropy production</u> in a fluid element

$$\sigma = -\frac{1}{T}\overset{\Rightarrow}{\Pi} : \text{grad}\,\vec{v} - \frac{1}{T^2}\vec{j}\cdot\text{grad}\,T$$

$$= +2\frac{\eta}{T}\left(e_{ij} - \frac{1}{3}\delta_{ij}\,\text{Tr}\,\overset{\Rightarrow}{e}\right)\left(e_{ij} - \frac{1}{3}\delta_{ij}\,\text{Tr}\,\overset{\Rightarrow}{e}\right)$$

$$+ \frac{\zeta}{T}\left(\text{div}\,\vec{v}\right)^2 + \frac{\lambda}{T^2}\left|\text{grad}\,T\right|^2 \qquad (2.3.4)$$

Using the second law of thermodynamics

$$\sigma \geqslant 0 \qquad (2.3.5)$$

one may conclude that

$$\eta \geqslant 0 \quad , \quad \zeta \geqslant 0 \quad \text{and} \quad \lambda \geqslant 0 \qquad (2.3.6)$$

The expression for the entropy production containing $\overset{\Rightarrow}{\Pi}$ and $\vec{j}$ is often used as a motivation of the relation (2.1.2) and (2.2.1). This method emphasizes the thermodynamic aspects of the problem. In this context grad $\vec{v}$ and grad T are interpreted as thermodynamic forces and $\overset{\Rightarrow}{\Pi}$ and $\vec{j}$ are the resulting thermodynamic current of momentum and heat; these are then usually assume to be linear functions of the thermodynamic forces which gives the relations (2.1.2) and (2.2.1) we used. In a simple fluid the thermodynamics is well understood and it does not really matter how eqs. (2.1.2) and (2.2.1) are found. In more complex systems as for instan liquid crystals it is wise to consider the problem from both angles to assure that on the one hand the correct thermodynamic variables have been used and on the other hand all the dissipative fluxes and forces have been found.

2.4 Thermal conduction

The description of the system is now given by the equation of continuity (1.1.5), the Navier-Stokes equations (2.1.3) and the equation for the internal energy (2.2.6)

$$\varrho \frac{Du}{Dt} = -p \operatorname{div}\vec{v} - \vec{\vec{\Pi}} : \operatorname{grad}\vec{v} - \operatorname{div}\vec{J}$$

$$= -p \operatorname{div}\vec{v} + 2\frac{\eta}{T}\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\, Tr\vec{\vec{e}}\right)\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\, Tr\vec{\vec{e}}\right) + \frac{\zeta}{T}(\operatorname{div}\vec{v})^2 + \lambda\Delta T \qquad (2.4.1$$

In this description the thermodynamic state of the system is given in terms of the density and the internal energy. The pressure and the temperature should be found using expressions $p(\varrho, u)$ and $T(\varrho, u)$. One may of course also describe the thermodynamic state of a fluid element in terms of the density and the temperature. The above equation must then be used to derive an equation for T. For this purpose we write

$$\frac{Du}{Dt} = \left(\frac{\partial u}{\partial T}\right)_\varrho \frac{DT}{Dt} + \left(\frac{\partial u}{\partial \varrho}\right)_T \frac{D\varrho}{Dt} \equiv c_V \frac{DT}{Dt} - \varrho\left(\frac{\partial u}{\partial \varrho}\right)_T \operatorname{div}\vec{v} \qquad (2.4.2$$

Here c_V is the specific heat at constant volume. Using the following thermodynamic relation

$$\varrho^2\left(\frac{\partial u}{\partial \varrho}\right)_T - p = -T\left(\frac{\partial p}{\partial T}\right)_\varrho \equiv -T\alpha \qquad (2.4.3$$

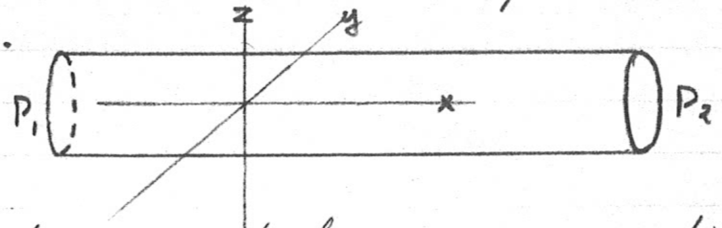
eq. (2.4.1) gives the following equation for the temperature

$$\varrho c_V \frac{DT}{Dt} = \lambda\Delta T - \alpha T \operatorname{div}\vec{v} + 2\frac{\eta}{T}\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\, Tr\vec{\vec{e}}\right)\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\, Tr\vec{\vec{e}}\right) + \frac{\zeta}{T}(\operatorname{div}\vec{v})^2 \qquad (2.4.4$$

The first term on the right hand side is the term which always appears in <u>Fourier's law</u>; the second term describes the change in the temperature due to adiabatic expansion or compression; the last two terms gives the temperature increase due to friction in pure strain flow and in isotropically expanding flow. The convective heat flux is on the left hand side of the equality. This convective flux is in many situation an important reason for temperature changes in particular because λ is usually rather small.

2.5 Poiseuille flow

As a special case we now consider incompressible steady flow in a cilindrical pipe.



The axis of the cilinder is chosen to be the x-axis. The length of the pipe is l. The current is caused by a pressure difference $p_1 - p_2 \equiv \delta p$. The equations to be solved are

$$\operatorname{div} \vec{v} = 0 \qquad \text{and } \operatorname{grad} p = \eta \Delta \vec{v} - \varrho\, \vec{v} \cdot \operatorname{grad} \vec{v} \qquad (2.5.1)$$

The velocity of the boundary is zero. Because of the translational and rotational symmetry of the problem we will try to solve these equations with a velocity of the form

$$\vec{v}(x,y,z) = (v(r), 0, 0) \qquad (2.5.2)$$

where $r^2 \equiv y^2 + z^2$. The reason to use no y or z component is in order to satisfy $\operatorname{div} \vec{v} = 0$. This choice also implies that the convective term is zero. We are therefore left with the equation, using the Laplace operator in cilindrical coordinates,

$$\operatorname{grad} p = \eta(\Delta v(r), 0, 0) = \eta\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} v(r), 0, 0\right) \qquad (2.5.3)$$

Clearly p depends only on and the equation

$$\frac{dp(x)}{dx} = \eta \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} v(r) \qquad (2.5.4)$$

The left hand side depends only on x and the right hand side only on r. Thus both sides are equal to a constant. This gives

$$p(x) = p_1 - \delta p \frac{x}{l} \qquad (2.5.5$$

The equation for v becomes

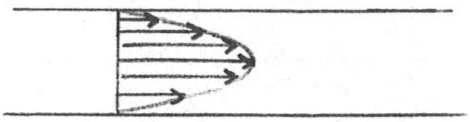
$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} v(r) = -\frac{\delta p}{\eta l} \qquad (2.5.6)$$

The general solution of this equation is

$$v(r) = -\frac{\delta p}{4\eta l} r^2 + a \ln r + b \qquad (2.5.7)$$

where a and b must be chosen such that e.g. the boundary conditions are satisfied. a must be zero in order to keep the solution finite. The solution which satisfies the boundary conditions is, R is the radius,

$$v(r) = \frac{\delta p}{4\eta l} (R^2 - r^2) \qquad (2.5.8)$$



This is the wellknown parabolic velocity profile due to Poiseuille. The mass Q which flows through the cilinder per second is

$$Q = 2\pi \rho \int_0^R r\, v(r)\, dr = \frac{\pi \rho \delta p}{8\eta l} R^4 \qquad (2.5.9)$$

<u>Pouseuille's formula</u>

This formula makes it possible to measure the viscosity by measuring the mass flow through a cilindrical pipe

<u>2.6 The law of similarity</u>

If one studies the motion of a viscous fluid a number of important results may be obtained from simple arguments concerning the dimensions of certain physical quantities. Let us consider a particular type of motion, the steady motion around a sphere, as an example. If one considers two different sizes of the sphere one has two problems which are <u>geometrically similar</u>. It is clear that the solution of the Navier-Stokes equation for both problems will be related if all variables are scaled properly. We shall discuss how this should be done. In order to make this as explicit as possible we shall consider the special case that the fluid is incompressible and furthermore $\vec{F} = 0$ first. The Navier-Stokes equation for steady flow for this case becomes

$$\vec{v} \cdot \operatorname{grad} \vec{v} = -\operatorname{grad}\left(\frac{p}{\rho}\right) + \nu \Delta \vec{v} \qquad (2.6.1)$$

The radius of the sphere is l. The other parameter which determines the nature of the flow which is under control in the experiment is the size of the velocity far from the sphere u. Note that the pressure has to be chosen such that $\mathrm{div}\,\vec{v}=0$ and is therefore not independently controlled. We now define

$$\vec{w} \equiv \vec{v}/u \qquad \text{and} \qquad \vec{s} \equiv \vec{r}/l \tag{2.6.2}$$

Both $\vec{w}$ and $\vec{s}$ are now dimensionless. Using these dimensionless quantities we may write eq. (2.6.1) as

$$\vec{w}\cdot\frac{\partial}{\partial\vec{s}}\vec{w} = -\frac{\partial}{\partial\vec{s}}\left(\frac{P}{\varrho u^2}\right) + \frac{\nu}{lu}\frac{\partial}{\partial\vec{s}}\cdot\frac{\partial}{\partial\vec{s}}\vec{w}$$

$$= -\frac{\partial}{\partial\vec{s}}\left(\frac{P}{\varrho u^2}\right) + \frac{1}{R}\frac{\partial}{\partial\vec{s}}\cdot\frac{\partial}{\partial\vec{s}}\vec{w} \tag{2.6.3}$$

where the <u>Reynolds number</u> is defined by

$$R \equiv \frac{lu}{\nu} = \frac{lu\varrho}{\eta} \tag{2.6.4}$$

It is clear from the above equations that the solution only depends on $\vec{s}$ and the Reynolds number. Thus

$$\vec{v}(\vec{r}) = u\,\vec{w}(\vec{s}, R) \qquad \underline{\text{law of similarity}} \tag{2.6.5}$$

Two flow problems with different values of u, l and ν but with the same value of R are called <u>similar</u>. In practice this law is extremely usefull It is sufficient to study the flow at different Reynolds numbers to know the behaviour for all values of l, u, ϱ and η.

If one also has gravity $|\vec{F}| = g$, the gravitational accelleration, one obtains a term in eq. (2.6.3) which has as prefactor one divided by

$$F \equiv \frac{u^2}{lg} \qquad \underline{\text{the Froude number}} \tag{2.6.6}$$

and the law of similarity becomes

$$\vec{v}(\vec{r}) = u\,\vec{w}(\vec{s}, R, F) \tag{2.6.7}$$

Now two flows are similar if both R and F are the same. Again this is very usefull. One may for instance study flow behaviour at a larger value of the gravitational constant by considering similar flow for which one increases instead l/u^2 keeping $u\,l$ (and thus R) constant. It is clear that this is usually much cheaper than changing g.

In the same way one may introduce many other such coefficients measuring essentially the relative importance of the various terms in the equations of motion for the fluid for all kind of experimental situations. The Reynolds number, which measures the importance of the convective momentum flow versu the viscous momentum flow, is the most common one. If R is small one may neglect the convective term which makes the Navier - Stokes equation linear in the velocity. This is clearly a dramatic simplificatio. The advantage is, however, that solutions are easier to construct. If R is large the viscous term may be neglected and the flow becomes ideal. The large R regime leads to very complicated flow phenomena as for instance turbulence. This kind of flow is very difficult to describe.

The flow behaviour for low Reynolds numbers is called <u>laminar</u>. Such laminar flow becomes unstable for larger values of the Reynolds number. If one considers e.g. the Poisseuille flow in a pipe one observes that, in spite of the fact that the solution which we constructed solves the equations for all values of the Reynolds number, the laminar flow breaks up at a critical value R_c of the Reynolds number and socalled turbulent eddies appear. The description of such transitions in non-linear dynamic systems is at present a hot research topic.

3 Formal solution for incompressible flow

3.1 Introduction

Velocity fields may in general be generated by applying forces on the fluid for instance by stirring a liquid with a spoon. The resulting motion is in general complicated. The reason for this is twofold. On the one hand is the force distribution on the surface of the spoon unknown, and owing to the shape rather complex. On the other hand, even if one would know the force distribution, it is in general impossible to construct a solution due to the convective contribution to the momentum flow. One way to improve this situation is to consider only low Reynolds number flow. In that case one may neglect the convective term which simplifies the situation considerably. The other difficulty, however, remains and as a result it is most common to solve the problem as far as possible using the boundary conditions. The fact that the surface of the spoon exerts a force on the fluid, and well such that the boundary condition is satisfied, is then not really important.

It must of course be admitted that whenever an explicit solution is possible there is very little to worry about. In many cases, however, such a solution does not exist and it becomes usefull to construct formal solutions. Even though a cynical person may say that formal solutions are nothing more than a reformulation of the problem, we shall see that many relevant questions may in fact be answered using these formal solutions, which express the velocity in terms of integrals over the force density.

We shall restrict the discussion in this chapter to incompressible flow. The general problem may be done in the same way; this is, however, so much more difficult that it is not worthwhile for these lectures. For many flow problems the incompressible fluid gives a proper description anyway and the general approach is much more transparant.

The equations describing incompressible flow are

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}.\text{grad}\,\vec{v} = -\text{grad}\frac{p}{\rho} + \nu\Delta\vec{v} + \vec{F} \qquad (3.1.1)$$

$$\text{div}\,\vec{v} = 0 \qquad (3.1.2)$$

Similar to the discussion in paragraph (1.6) one may replace the second equation by an equation for the pressure which is obtained by taking the divergence of the first equation. This gives again eq. (1.6.3) being

$$\Delta\left(\frac{p}{\rho}\right) = \vec{\nabla}.\left[\vec{F} - \vec{v}.\vec{\nabla}\vec{v}\right] = \vec{\nabla}.\vec{F} - (\vec{\nabla}\vec{v}):(\vec{\nabla}\vec{v}) \qquad (3.1.3)$$

For the construction of the formal solution it is convenient to define

$$\vec{G} \equiv \vec{F} - \vec{v}.\text{grad}\,\vec{v} \qquad (3.1.4$$

This combined "force" contains all the terms which make the construction of an explicit solution difficult. We now write eqs. (3.1.1) and (3.1.3) in the following form

$$\frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} = \left(\frac{\partial}{\partial t} - \nu\Delta\right)\vec{v} = -\text{grad}\frac{p}{\rho} + \vec{G} \qquad (3.1.5)$$

$$\Delta\left(\frac{p}{\rho}\right) = \vec{\nabla}.\vec{G} \qquad (3.1.6)$$

The formal solution, which we want to obtain, expresses $\vec{v}$ and p in terms of $\vec{G}$.

Using Fourier transformation methods this is in fact not a difficult problem. It is instructive, however, to first consider the steady flow case for which one may construct these solutions without Fourier transformation. We will then later consider the general case.

3.2 Steady flow

For steady flow the equation (3.1.5) reduces to

$$\Delta\vec{v} = \frac{1}{\eta}\,\text{grad}\,p - \frac{1}{\nu}\vec{G} \qquad (3.2.1)$$

which must be solved together with equation (3.1.6) for the pressure. In order to construct the general solution we first consider the following special case where the force density is a point source in $\vec{r}=0$

$$\vec{G}(\vec{r}) = 8\pi\nu\vec{\alpha}\,\delta(\vec{r}) \tag{3.2.2}$$

The equation for the pressure now becomes

$$\Delta p = 8\pi\eta\vec{\alpha}\cdot\vec{\nabla}\delta(\vec{r}) \tag{3.2.3}$$

This equation may immediately be solved using

$$\Delta\frac{1}{r} = -4\pi\delta(\vec{r}) \implies \Delta\left(\vec{\alpha}\cdot\vec{\nabla}\frac{1}{r}\right) = (\vec{\alpha}\cdot\vec{\nabla})\Delta\frac{1}{r} = -4\pi\vec{\alpha}\cdot\vec{\nabla}\delta(\vec{r})$$

We thus find

$$p = -2\eta\vec{\alpha}\cdot\vec{\nabla}\frac{1}{r} = 2\eta\frac{\vec{\alpha}\cdot\hat{r}}{r^2} \tag{3.2.4}$$

where we have used a zero value of p in infinity. Notice that we construct solutions as if the fluid fills all of space without boundaries. The reason that we may do this is that the forces on the boundaries which keep the fluid inside are all contained in $\vec{G}$. Using this $\vec{G}$ one should then construct solutions in infinite space. In order to find the velocity field it is convenient to first calculate the vorticity. For this purpose we use the identity

$$\text{rot rot} = \text{grad div} - \Delta \tag{3.2.5}$$

Using the fact that $\text{div}\,\vec{v}=0$ and $\text{rot}\,\vec{v}=\vec{\omega}$ we find from equation (3.2.1)

$$\text{rot}\,\vec{\omega} = -\frac{1}{\eta}\,\text{grad}\,p + 8\pi\vec{\alpha}\cdot\delta(\vec{r}) \tag{3.2.6}$$

Taking the rotation of this equation using eq. (3.2.5) and $\text{div}\,\vec{\omega}=0$ we obtain

$$\Delta \vec{\omega} = 8\pi \vec{\alpha} \wedge \vec{\nabla} \delta(\vec{r}) \tag{3.2.7}$$

The solution of this equation is again simple to construct

$$\vec{\omega} = -2\vec{\alpha} \wedge \vec{\nabla} \frac{1}{r} = \frac{2}{r^2}(\vec{\alpha} \wedge \hat{r}) \tag{3.2.8}$$

The expression for the velocity is found using $\vec{\omega} = \mathrm{rot}\,\vec{v}$
One may easily verify that

$$\vec{v} = \frac{1}{r} \vec{\alpha} \cdot (\vec{\vec{1}} + \hat{r}\hat{r}) \tag{3.2.9}$$

If one verifies this, it is convenient to use the following two derivatives

$$\vec{\nabla} f(r) = \hat{r} \frac{\partial}{\partial r} f(r) \qquad \text{and} \qquad \vec{\nabla} \vec{r} = \vec{\vec{1}} \tag{3.2.10}$$

where f is an arbitrary function of the absolute value $r \equiv |\vec{r}|$.

To sum it all up we find:

$$\vec{g}(\vec{r}) = 8\pi \nu \vec{\alpha}\, \delta(\vec{r}) \Longrightarrow \begin{cases} \vec{v}(\vec{r}) = \frac{1}{r} \vec{\alpha} \cdot (\vec{\vec{1}} + \hat{r}\hat{r}) \\ p(\vec{r}) = 2\eta \frac{1}{r^2} \vec{\alpha} \cdot \hat{r} \\ \vec{\omega}(\vec{r}) = \frac{2}{r^2} (\vec{\alpha} \wedge \hat{r}) \end{cases} \tag{3.2.11}$$

The source is called a <u>Stokeslet</u> with strength $\vec{\alpha}$. The position of this Stokeslet is the origin of the coordinate frame one may of course have Stokeslets at arbitrary positions in space.

In view of the fact that by construction $\vec{v}$, p and w are linear in $\vec{g}$ one may now write the general solution of eqs. (3.1.6) and (3.2.1) as

$$\vec{v}(\vec{r}) = \frac{1}{8\pi\nu} \int d\vec{r}\,' \, \vec{g}(\vec{r}\,') \cdot \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(\vec{\vec{1}} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \right) \right] \tag{3.2.12}$$

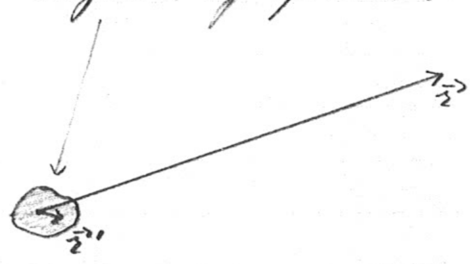
$$p(\vec{r}) = \frac{\rho}{4\pi} \int d\vec{r}\,' \, \vec{g}(\vec{r}\,') \cdot \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \tag{3.2.13}$$

$$\vec{w}(\vec{r}) = \frac{1}{4\pi\nu}\int d\vec{r}'\; \vec{G}(\vec{r}') \wedge \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \qquad (3.2.14)$$

Important to remember is that the velocity field due to a Stokeslet, or in other words due to a point force, goes to zero as one over the distance for large distances. This is slow in particular if one realises that the amount of space available increases as the distance squared. The hydrodynamic "interaction" between different parts of the fluid is therefore long range

3.3 Stokes multipoles

In many cases one has a force distribution which is only unequal to zero in a small region of space. If one measures the velocity in $\vec{r}$ far from this region it is clear that the precize shape of the region is not really important



The main contribution is simply due to the total force neglecting the size of the region or, if the total force is zero, due to the lowest order moment which is unequal to zero.

We shall now proceed to formulate this in a more precise fashion. Take the center of the coordinate system in the small region. We may then use $|\vec{r}'| \ll |\vec{r}|$ in the expressions for $\vec{v}(\vec{r})$, $p(\vec{r})$ and $w(\vec{r})$ given in the previous paragraph. Performing a Taylor expansion of a function $f(\vec{r}')$ is done using the formula

$$f(\vec{r}') = f(0) + \vec{r}' \cdot \left[\frac{\partial}{\partial \vec{r}'} f(\vec{r}')\right]_{\vec{r}'=0} + \frac{1}{2}\, \vec{r}'\vec{r}' : \left[\frac{\partial}{\partial \vec{r}'}\frac{\partial}{\partial \vec{r}'} f(\vec{r}')\right]_{\vec{r}'=0} + \cdots \qquad (3.3.1)$$

Expansion of the integral kernel in the expression for the velocity gives

$$\frac{1}{|\vec{r}-\vec{r}'|}\left(\vec{\vec{1}} + \frac{(\vec{r}-\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}\right)_{ij} = \frac{1}{r}(\delta_{ij} + \hat{r}_i\hat{r}_j) - r'_k \frac{\partial}{\partial r_k}\left[\frac{1}{r}(\delta_{ij} + \hat{r}_i\hat{r}_j)\right]$$

$$+ \frac{1}{2} r'_k r'_l \frac{\partial}{\partial r_k}\frac{\partial}{\partial r_l}\left[\frac{1}{r}(\delta_{ij} + \hat{r}_i\hat{r}_j)\right] + \cdots \qquad (3.3.2)$$

Before we proceed to calculate the derivatives of the Stokeslet field we substitute this expression in equation (3.2.12) for the velocity field which gives

$$v_i(\vec{r}) = \frac{1}{r}\left(\delta_{ij} + \hat{r}_i \hat{r}_j\right) \frac{1}{8\pi\nu} \int d\vec{r}\,' \, G_j(\vec{r}\,')$$
$$-\left[\frac{\partial}{\partial r_k} \frac{1}{r}\left(\delta_{ij} + \hat{r}_i \hat{r}_j\right)\right] \frac{1}{8\pi\nu} \int d\vec{r}\,' \, r_k' \, G_j(\vec{r}\,')$$
$$+\left[\frac{\partial}{\partial r_k}\frac{\partial}{\partial r_\ell} \frac{1}{r}\left(\delta_{ij} + \hat{r}_i \hat{r}_j\right)\right] \frac{1}{16\pi\nu} \int d\vec{r}\,' \, r_k' \, r_\ell' \, G_j(\vec{r}\,') + \cdots \qquad (3.3.3)$$

The total <u>Stokeslet strength</u> is

$$\vec{\alpha} \equiv \frac{1}{8\pi\nu} \int d\vec{r}\,' \, \vec{G}(\vec{r}\,') \qquad (3.3.4)$$

The <u>Stokes dipole strength</u> is

$$\overset{\Rightarrow}{\alpha}_D \equiv \frac{1}{8\pi\nu} \int d\vec{r}\,' \quad \vec{G}(\vec{r}\,') \, \vec{r}\,' \qquad (3.3.5)$$

and the <u>Stokes quadrupole strength</u> is

$$\overset{\Rrightarrow}{\alpha}_Q \equiv \frac{1}{16\pi\nu} \int d\vec{r}\,' \, \vec{G}(\vec{r}\,') \, \vec{r}\,' \vec{r}\,' \qquad (3.3.6)$$

In the same way one may proceed to define multipoles of arbitrary order. Using these multipoles one may now write the velocity field as

$$\vec{v}(\vec{r}) = \vec{\alpha} \cdot \left[\frac{1}{r}\left(\overset{\Rightarrow}{1} + \hat{r}\hat{r}\right)\right] - \overset{\Rightarrow}{\alpha}_D : \left[\vec{\nabla} \frac{1}{r}\left(\overset{\Rightarrow}{1} + \hat{r}\hat{r}\right)\right]$$
$$+ \overset{\Rrightarrow}{\alpha}_Q \vdots \left[\vec{\nabla}\vec{\nabla} \frac{1}{r}\left(\overset{\Rightarrow}{1} + \hat{r}\hat{r}\right)\right] + \cdots \qquad (3.3.7)$$

The expressions between the square brackets give respectively the velocity field of a Stokeslet, a Stokes dipole and a Stokes quadrupole.

We shall now first proceed to consider the field due to a Stokes dipole $\overset{\Rightarrow}{\alpha}_D$ in more detail. One has

$$\frac{\partial}{\partial r_i} \frac{1}{r}\left(\delta_{jk} + \hat{r}_j \hat{r}_k\right) = \frac{1}{r^2}\left(\delta_{ij}\hat{r}_k + \delta_{ik}\hat{r}_j - \delta_{jk}\hat{r}_i - 3\hat{r}_i\hat{r}_j\hat{r}_k\right) \qquad (3.3.8)$$

note that $\Delta \left[3 \overset{\Rightarrow}{\alpha}{}^{s}_{D} : \frac{\hat{r}\hat{r}\hat{r}}{r^2} \right] = \overset{\Rightarrow}{\alpha}{}^{s}_{D} : \left[12 \frac{\hat{r}}{r^4} \left(\overset{\Rightarrow}{I} - \frac{5}{2} \hat{r}\hat{r} \right) \right]$

The δ-function contributions may most conveniently be found by volume integration over a small sphere around zero

$$\int_V p \, d\vec{r} = -2\eta \, \alpha^s_{Dij} \int_V \frac{\partial}{\partial r_j} \frac{\hat{r}_i}{r^2} \, d\vec{r} = -2\eta \, \alpha^s_{Dij} \int_S \hat{r}_j \hat{r}_i \frac{dS}{r^2}$$

$$= -\frac{8\pi}{3} \eta \, \alpha^s_{Dij} \, \delta_{ij} = -\frac{8\pi}{3} \eta \, T_r \overset{\Rightarrow}{\alpha}{}^{s}_{D}$$

which identifies the coefficient of the δ-function. One might of course also use

$$\vec{\nabla} \frac{\hat{r}}{r^2} = - \vec{\nabla}\vec{\nabla} \frac{1}{r} = \frac{4\pi}{3} \delta(\vec{r}) + \frac{1}{r^3} \left(\overset{\Rightarrow}{I} - 3 \hat{r}\hat{r} \right)$$

which gives the same result. This method is, however, often less transparant. If on for instance calculates $\vec{\omega}$ by differentiating $\vec{v}$ directly it is not at all clear why no δ-function contribution appears. Using the first method gives

$$\int_V \omega_k \, d\vec{r} = -2 \, \epsilon_{kij} \, \alpha^s_{Dil} \int_V \frac{\partial}{\partial r_l} \frac{\hat{r}_j}{r^2} \, d\vec{r}$$

$$= -2 \, \epsilon_{kij} \, \alpha^s_{Dil} \int_S \hat{r}_l \hat{r}_j \frac{dS}{r^2} = -\frac{8\pi}{3} \epsilon_{kij} \, \alpha^s_{Dil} \, \delta_{lj}$$

$$= -\frac{8\pi}{3} \epsilon_{kij} \, \alpha^s_{Dij} \qquad = 0$$

because ϵ is antisymmetric in i,j while α^s_D is symmetric. One may also differentiate directly in

$$\vec{\omega} = -2 \left(\overset{\Rightarrow}{\alpha}{}^{s}_{D} \cdot \vec{\nabla} \right) \wedge \frac{\hat{r}}{r^2}$$

which also gives the correct result.

One may now write $\vec{\vec{\alpha}}_D$ as a sum of a symmetric and an antisymmetric part

$$\vec{\vec{\alpha}}_D = \vec{\vec{\alpha}}_D^{\,S} + \vec{\vec{\alpha}}_D^{\,A} \qquad (3.3.9)$$

Using eq. (3.3.8) one has

$$\vec{\vec{\alpha}}_D : \left[\vec{\nabla}\frac{1}{r}\left(\vec{\vec{1}} + \hat{r}\hat{r}\right)\right] = \left(\alpha^S_{Dji} + \alpha^A_{Dji}\right)\frac{1}{r^2}\left(\delta_{ij}\hat{r}_k + \delta_{ik}\hat{r}_j - \delta_{jk}\hat{r}_i - 3\hat{r}_i\hat{r}_j\hat{r}_k\right)$$

$$= \alpha^S_{Dji}\left[\frac{1}{r^2}\left(\delta_{ij} - 3\hat{r}_i\hat{r}_j\right)\hat{r}_k\right] + \alpha^A_{Dji}\left[\frac{1}{r^2}\left(\delta_{ik}\hat{r}_j - \delta_{jk}\hat{r}_i\right)\right] \qquad (3.3.10)$$

The symmetric part of a Stokes dipole $\vec{\vec{\alpha}}_D^{\,S}$ is called a <u>stresslet</u>. The velocity field due to a stresslet has been constructed above. The pressure and the vorticity are found similarly

$$\vec{V}(\vec{r}) = -\vec{\vec{\alpha}}_D^{\,S} : \left(\vec{\vec{1}} - 3\hat{r}\hat{r}\right)\frac{\hat{r}}{r^2} = 3\,\vec{\vec{\alpha}}_D^{\,S} : \frac{\hat{r}\hat{r}\hat{r}}{r^2}$$

$$p(\vec{r}) = -2\eta\,\vec{\vec{\alpha}}_D^{\,S} : \vec{\nabla}\frac{\hat{r}}{r^2} = -\frac{8\pi}{3}\eta\left(\mathrm{Tr}\,\vec{\vec{\alpha}}_D^{\,S}\right)\delta(\vec{r}) - 2\eta\,\vec{\vec{\alpha}}_D^{\,S} : \frac{1}{r^3}\left(\vec{\vec{1}} - 3\hat{r}\hat{r}\right)$$

$$\vec{\omega}(\vec{r}) = -2\left(\vec{\vec{\alpha}}_D^{\,S}\cdot\vec{\nabla}\right)\wedge\frac{\hat{r}}{r^2} = 6\left(\vec{\vec{\alpha}}_D^{\,S}\cdot\hat{r}\right)\wedge\frac{\hat{r}}{r^3} \qquad (3.3.11)$$

The δ-function in the pressure is necessary to make these fields solutions in $\vec{r}=0$ with the proper choice of the stresslet source.

The antisymmetric part of a tensor may be written in the following form

$$\alpha^A_{Dji} = -\frac{1}{2}\varepsilon_{jil}\,\gamma_l \qquad (3.3.12)$$

where $\vec{\gamma}$ is a vector. One then finds

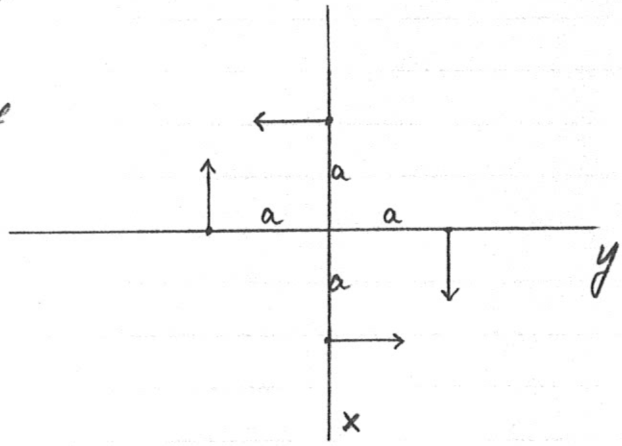
$$\alpha^A_{Dji}\left[\frac{1}{r^2}\left(\delta_{ik}\hat{r}_j - \delta_{jk}\hat{r}_i\right)\right] = -\left(\vec{\gamma}\wedge\frac{\hat{r}}{r^2}\right)_k \qquad (3.3.13)$$

The antisymmetric part of of a Stokes dipole $\vec{\vec{\alpha}}_D^{\,A}$ is called a <u>rotlet</u> or a <u>couplet</u>. Its velocity, pressure and vorticity field are given by

$$\vec{V}(\vec{r}) = \vec{\gamma}\wedge\frac{\hat{r}}{r^2}, \quad p(\vec{r}) = 0$$

$$\vec{\omega}(\vec{r}) = \mathrm{rot}\left(\vec{\gamma}\wedge\frac{\hat{r}}{r^2}\right) = \mathrm{rot}\,\mathrm{rot}\left(\frac{\vec{\gamma}}{r}\right) = 4\pi\vec{\gamma}\,\delta(\vec{r}) - \vec{\gamma}\cdot\left(\vec{\vec{1}} - 3\hat{r}\hat{r}\right)\frac{1}{r^3} \qquad (3.3.14)$$

For a Stokeslet we know the source which is a point force. One may construct similar point sources for a stresslet and a rotlet. To make this somewhat more transparant we will consider a force distribution which gives a stresslet. In the figure such a force distribution is drawn. The absolute value of the forces is equal to K. The strength of the Stokes dipole is given by



$$\vec{\vec{\alpha}}_D = \frac{aK}{4\pi\nu}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \vec{\vec{\alpha}}_D^{\,s} \qquad (3.3.15)$$

This is clearly symmetric. It is also clear from the force distribution that <u>stress lets give pure straining flow</u>. The force distribution may be written down as

$$\vec{G}(\vec{r}) = K(0,1,0)\left[\delta(\vec{r}-a(1,0,0)) - \delta(\vec{r}+a(1,0,0))\right]$$

$$+ K(1,0,0)\left[\delta(\vec{r}-a(0,1,0)) - \delta(\vec{r}+a(0,1,0))\right] \qquad (3.3.16$$

Just as in the construction of an electric dipole using two point charges one may now take the limit $a \to 0$, $K \to \infty$ with aK constant. In that case we may Taylor expand the δ-functions in a and keep only the linear term

$$\vec{G}(\vec{r}) = -2Ka\left[(0,1,0)(1,0,0) + (1,0,0)(0,1,0)\right]\cdot\vec{\nabla}\,\delta(\vec{r})$$

$$= -8\pi\nu\,\vec{\vec{\alpha}}_D^{\,s}\cdot\vec{\nabla}\,\delta(\vec{r}) \qquad (3.3.1$$

Thus we find that the <u>force distribution for a stresslet</u> is proportional to the derivative of a δ-function. One may easily verify that this force distribution gives the fields due to a stresslet. We have for instance for the velocity, cf. eq. (3.2.12),

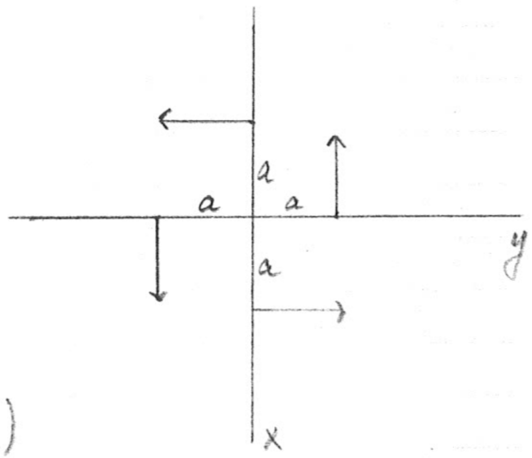
$$\vec{v}(\vec{r}) = -\int d\vec{r}\,'\left(\vec{\vec{\alpha}}_D^{\,s}\cdot\vec{\nabla}'\delta(\vec{r}\,')\right)\cdot\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{I}} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right]\right.$$

$$= \int d\vec{r}\,' \, \overset{\Rightarrow}{\alpha}{}^{s}_{D} : \delta(\vec{r}\,') \, \vec{\nabla}' \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(\overset{\Rightarrow}{I} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \right) \right]$$

$$= \overset{\Rightarrow}{\alpha}{}^{s}_{D} : \vec{\nabla}' \left[\quad \text{same} \quad \right]_{\vec{r}\,'=0} = - \overset{\Rightarrow}{\alpha}{}^{s}_{D} : \vec{\nabla} \left[\frac{1}{r} \left(1 + \hat{r}\hat{r} \right) \right]$$

This is exactly the velocity field due to a Stokes dipole given as second term in equation (3.3.7) which if $\overset{\Rightarrow}{\alpha}_D$ is symmetric is the velocity field due to a stresslet.

For the rotlet the force distribution is the one in the adjoining figure. Now we have



$$\overset{\Rightarrow}{\alpha}_D = \frac{aK}{4\pi\nu} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \overset{\Rightarrow}{\alpha}{}^{A}_{D} \qquad (3.3.18)$$

This gives for $\vec{\gamma}$, cf. eq. (3.3.12),

$$\vec{\gamma} = \frac{aK}{2\pi\nu} (0, 0, 1) \qquad (3.3.19)$$

In a similar way as for the stresslet we find as <u>force density for the rotlet</u> (or couplet)

$$\vec{g}(\vec{r}) = -8\pi\nu \, \overset{\Rightarrow}{\alpha}{}^{A}_{D} \cdot \vec{\nabla} \delta(\vec{r}) = -4\pi\nu \, \vec{\gamma} \wedge \vec{\nabla} \delta(\vec{r}) \qquad (3.3.20)$$

One may again easily verify that this force density gives the velocity and other fields due to a rotlet. A rotlet gives purely rotational flow. The rotational velocity, however, goes to zero as $1/r^2$ and thus the fluid does not rotate as a rigid body. The inner layers flow faster.

If one calculates the total force on the fluid

$$\vec{K} \equiv \int d\vec{r} \, \vec{g}(\vec{r}) \qquad (3.3.21)$$

One finds for the Stokeslet, stresslet and rotlet

$$\vec{K}_{\text{stokeslet}} = 8\pi\nu\vec{\alpha} \quad \text{and} \quad \vec{K}_{\text{stresslet}} = \vec{K}_{\text{rotlet}} = 0 \qquad (3.3.22)$$

The stresslet and the rotlet give no net force on the fluid. If one calculates the total torque on the fluid

$$\vec{M} \equiv \int \vec{r} \wedge \vec{G}(\vec{r})\, d\vec{r} \qquad (3.3.23)$$

one finds that the Stokeslet and the stresslet do not give a net torque on the fluid

$$\vec{M}_{stokeslet} = \vec{M}_{stresslet} = 0 \qquad (3.3.24)$$

For the rotlet we find, however,

$$\vec{M}_{rotlet} = -4\pi\nu \int \vec{r} \wedge \left(\vec{\gamma} \wedge \vec{\nabla}\delta(\vec{r})\right) d\vec{r} = 8\pi\nu\vec{\gamma} \qquad (3.3.25)$$

as relation between the total torque on the fluid and $\vec{\gamma}$

We have now discussed the fields due to the Stokes monopole, the Stokeslet, and the Stokes dipoles, the stresslet and the rotlet. We must stil consider the Stokes quadrupole fields. It is clear that the <u>general quadrupole field</u> is generated by a <u>force density</u> proportional to the second derivative of the δ-function

$$\vec{G}(\vec{r}) = 16\pi\nu\, \overset{\Rightarrow}{\vec{\alpha}}_Q : \vec{\nabla}\vec{\nabla}\delta(\vec{r}) \qquad (3.3.26)$$

Substitution of this force density in eq. (3.2.12) gives the quadrupole contribution in eq. (3.3.7). One may of course again consider symmetric and anti-symmetric combinations (with respect to the first two indices as $\overset{\Rightarrow}{\vec{\alpha}}_Q$ may always be chosen symmetric in the last two indices). We shall not try to do this as this would carry us to far. One particular contribution, which is of particular interest, we shall discuss, however. Choose

$$\alpha_{Q,ijk} = \beta_i\, \delta_{jk} \qquad (3.3.27)$$

This is called a <u>potential doublet</u>. The <u>force density</u> for such a <u>potential doublet</u> is given by

$$\vec{G}(\vec{r}) = 8\pi\nu\vec{\beta}\, \Delta\delta(\vec{r}) \qquad (3.3.28)$$

Using this force density one may find $\vec{v}$, p and $\vec{\omega}$ by substitution in eqs. (3.2.12) - (3.2.14). This gives

$$\vec{v}(\vec{r}) = -\frac{16\pi}{3}\vec{\beta}\,\delta(\vec{r}) + 2\vec{\beta}\cdot\frac{1}{r^3}(1-3\hat{r}\hat{r})$$

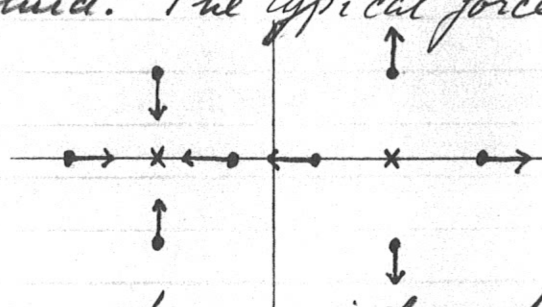
$$p(\vec{r}) = 8\pi\eta\vec{\beta}\cdot\vec{\nabla}\delta(\vec{r})$$

$$\vec{\omega}(\vec{r}) = 8\pi\vec{\beta}\wedge\vec{\nabla}\delta(\vec{r}) \qquad (3.3.29)$$

In this case only the velocity field is unequal to zero for $\vec{r}\neq 0$. The velocity field for $\vec{r}$ unequal to zero is equal to the potential flow used to solve the flow around a sphere in the ideal fluid in paragraph (1.8). The potential doublet is the typical source of potential flow. Just like the Stokeslet can be used to construct by linear superposition more general flows one may use the potential doublet to construct by linear superposition general potential flows. The above analysis shows however that the potential doublet is itself a combination of Stokelets. The total force and the total torque on the fluid due to a potential doublet are zero

$$\vec{K}_{pot.doublet} = \vec{M}_{pot.\ doublet} = 0 \qquad (3.3.30)$$

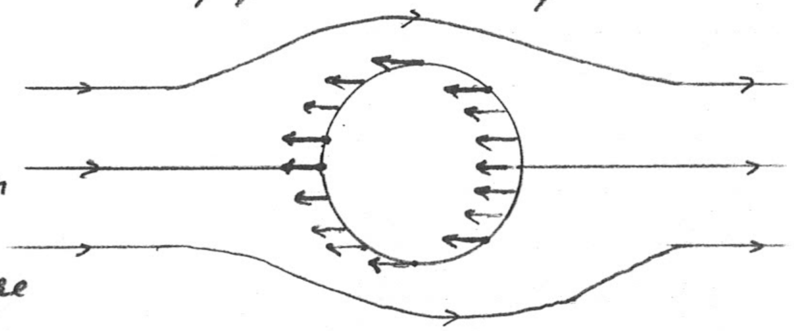
Thus if one has potential flow there is in general no total force or torque on the fluid. The typical force distribution which leads to a potential doublet is drawn in the figure. In the limit of small distances and large forces such $\vec{\beta}$ is constant the fluid seems, so to say, disappear at one point and reappear at another (both indicated by crosses).



One final note of warning should be made. By using a particular choice of $\vec{g}$ it is possible to construct explicit solutions of the Navier-Stokes equation. For a problem where the Reynolds number is not small the $(-\vec{v}\cdot\text{grad}\,\vec{v})$ contribution to $\vec{g}$ becomes important and the solution as well as the force distribution $\vec{g}$ must be determined selfconsistently. This is clearly a very difficult procedure but nevertheless usefull for certain purposes.

3.4 Stokes flow around a sphere

We consider again the flow around a sphere in an incompressible fluid. In paragraph 1.8 we analysed this problem for an ideal fluid. Now we consider the more general case with a finite viscosity. We restrict ourselves to low Reynolds numbers so that the viscous contribution to the pressure tensor is important compared to the conductive contribution ($\rho\vec{v}$ Neglecting the conductive contribution one has $\vec{G} = \vec{F}$. The force is thus only unequal to zero on the surface of the sphere and should be chosen such that the resulting flow due to this force distribution, when added to the incident velocity field satisfies the boundary condition



The force distribution should look more or less the way it is drawn in the figure. Far away from the sphere the velocity is $\vec{u}$. The center of the coordinate frame is chosen to coincide wit the center of the sphere.

The force distribution may be expanded in a multipole expansion Because of the rotational symmetry of the sphere the strength of the resulting multipoles can only depend on the vector $\vec{u}$. For low Reynolds numbers the problem is linear and thus the strength of the multipoles is necessarily linear in $\vec{u}$ We may therefore conclude that the stresslet strengt is zero as it is impossible to make a tensor which is a linear function of $\vec{u}$. The rotlet strength must also be zero because if $\vec{\gamma} \sim \vec{u}$ one would have rotation of the fluid around $\vec{u}$. This is impossible because of reflection symmetry with respect to any plane parallel to $\vec{u}$ through the center of the sphere. Thus only the Stokeslet and the potential doublet strength remai Of course higher order multipoles might also give fields proportional to $\vec{u}$ in principle, but in this case they will not be needed as we shall see. We consequently use the following force distribution

$$\vec{g}(\vec{r}) = 8\pi\nu\left[c_1 \vec{u}\,\delta(\vec{r}) + c_2 \vec{u}\,\Delta\delta(\vec{r})\right] \qquad (3.4.1)$$

The incident velocity field plus the velocity field due to $\vec{g}$ is

$$\vec{v}(\vec{r}) = \vec{u} + \frac{c_1}{r}\vec{u}\cdot(\vec{\vec{1}} + \hat{r}\hat{r}) + 2\frac{c_2}{r^3}\vec{u}\cdot(\vec{\vec{1}} - 3\hat{r}\hat{r}) \qquad \text{for } r > 0 \qquad (3.4.2)$$

In order to satisfy the boundary condition $\vec{v}(\vec{r})$ must be zero for $r = a$ on the surface of the sphere. Here a is the radius of the sphere. It follows from the above expression that this is the case if

$$\left.\begin{array}{l} 1 + \dfrac{c_1}{a} + 2\dfrac{c_2}{a^3} = 0 \\ \dfrac{c_1}{a} - 6\dfrac{c_2}{a^3} = 0 \end{array}\right\} \Rightarrow c_1 = -\tfrac{3}{4}a \text{ and } c_2 = -\tfrac{1}{8}a^3 \qquad (3.4.3)$$

The velocity field is thus given by

$$\vec{v}(\vec{r}) = \vec{u} - \frac{3}{4}\left(\frac{a}{r}\right)\vec{u}\cdot(\vec{\vec{1}} + \hat{r}\hat{r}) - \frac{1}{4}\left(\frac{a}{r}\right)^3\vec{u}\cdot(\vec{\vec{1}} - 3\hat{r}\hat{r}) \qquad \text{for } r > a \qquad (3.4.3)$$

This is the expression given first by Stokes for incompressible steady flow around a sphere at low Reynolds numbers.

The pressure may also easily be given using the fact that the potential doublet does not contribute and using eq. (3.2.11)

$$p(\vec{r}) = -\frac{3}{2}\frac{\eta}{a}\left(\frac{a}{r}\right)^2 \vec{u}\cdot\hat{r} \qquad \text{for } r > a \qquad (3.4.4)$$

The pressure at infinity was chosen to be zero. Note the fact that the pressure at stagnation points is no longer equal as we found for the ideal fluid. In the present case the pressure is actually larger on the side where the fluid is incident.

The total force due to the sphere is also due only to the Stokeslet strength and we thus find, cf. eq. (3.3.22),

$$\vec{K} = -6\pi\nu\vec{u}\,a \qquad (3.4.5)$$

as the total force (per unit of mass) exerted by the sphere on the fluid. As a consequence the force exerted by the fluid on the sphere (action is minus reaction) is given by

$$\vec{H} = 6\pi\eta\vec{u}a \qquad \underline{\text{Stokes law}} \qquad (3.4.6)$$

As this force should not be calculated per unit of fluid mass we had to multiply $\vec{K}$ by minus ϱ! In the more general case that the velocity of the sphere is finite one finds as force on the sphere, using Galilean invariance,

$$\vec{H} = 6\pi\eta(\vec{u}-\vec{w})a \qquad (3.4.7)$$

where $\vec{w}$ is the velocity of the sphere. The motion of a spherical inclusion in a fluid is thus governed by Stokes law.

It is interesting to use the solutions given above in order to calculate the real force density. According to eq. (3.2.1) we have

$$\vec{G} = \frac{1}{\varrho}\,\mathrm{grad}\,p - \nu\Delta\vec{v} \qquad (3.4.8)$$

Clearly $\vec{G}$ is only unequal to zero only on the surface of the sphere. Inside the sphere we have $p=0$ and $\vec{v}=0$. Because of the fact that p is discontinuous on the surface of the sphere the gradient gives a δ function on the surface. The velocity is continuous; its gradient is discontinuous and the divergence of the gradient gives also a δ function on the surface. In this way one may, by straightforward differentiation, show that

$$\vec{G}(\vec{r}) = -\frac{3}{2a}\nu\vec{u}\cdot\hat{r}\hat{r}\,\delta(r-a) - \frac{3}{2a}\nu\vec{u}\cdot(\vec{1}-\hat{r}\hat{r})\,\delta(r-a) = -\frac{3\nu\vec{u}}{2a}\,\delta(r-a) \qquad (3.4.9)$$

where the first term is due to the pressure and the second to the velocity. If one uses this force distribution to calculate $\vec{v}$ and p from eqs. (3.2.12) and (3.2.13) one finds the Stokes fields, eqs (3.4.3) and (3.4.4), outside the sphere and $\vec{v}=0$, $p=0$ inside the sphere as one should find from the exact force distribution.

3.5 Faxén's theorem

We again consider stationary flow around a sphere with fixed position. Using Stokes law one may then in principle measure the viscosity in uniform flow considered in the previous section. A possible method is to have the fluid flow upwards in a pipe. A spherical particle which is heavier than the fluid will then be pulled down by gravity and pushed up by Stokes. The trick is to adjust the fluid velocity such that the position of the sphere does not change. The problem which bothered Faxén was that, even when the size of the pipe was rather much bigger than the diameter of the sphere, the resulting viscosity depended on the diameter of the pipe. The reason is obviously the fact that, due to the boundaries, the fluid velocity is not homogeneous (Poisseuille flow). Faxén then derived an ingeneous theorem which made it possible to calculate the viscosity without this difficulty.

Faxén defined as a first step the velocity field $\vec{v}_0$ which one has in the absence of the sphere. This velocity field is caused by the forces pumping the fluid through the system and contains the effects due to the resulting induced forces on the surface of the container which are necessary to satisfy the boundary conditions. It does not however contain the modifications due to the sphere. In the case of Stokes flow $\vec{v}_0(\vec{r}) = \vec{u}$ while in a pipe $\vec{v}_0(\vec{r})$ is given by the Poisseuille flow profile. As for Stokes flow Faxén restricted himself to low Reynolds numbers so that $\vec{G} = \vec{F}$. We may write

$$\vec{G} = \vec{G}_{sphere} + \vec{G}_{rest\,of\,system} \qquad (3.5.1)$$

The second term gives $\vec{v}_0$ and the first the modification due to the sphere. Using eq. (3.2.12) we may thus write

$$\vec{v}(\vec{r}) = \vec{v}_0(\vec{r}) + \frac{1}{8\pi\nu}\int d\vec{r}\,'\,\vec{G}_s(\vec{r}\,')\cdot\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{1}} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right)\right] \qquad (3.5.2)$$

We know that $\vec{v}(\vec{r})$ is zero on the surface of the sphere. The trick to obtain Faxén's theorem is to average the above equation for $\vec{v}(\vec{r})$ over the surface of the sphere.

We first calculate the following average

$$\frac{1}{4\pi a^2}\int dS\,\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{1}}+\frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right) = c_1(r')\,\vec{\vec{1}} + c_2(r')\,\hat{r}'\hat{r}' \qquad (3.5.3)$$

As after averaging the only remaining vector is $\vec{r}\,'$ it is clear that the average must have the form given on the right. Taking the trace we have

$$3c_1+c_2 = \frac{1}{\pi a^2}\int dS\,\frac{1}{|\vec{r}-\vec{r}\,'|} = 2\int_0^{\pi} d\theta\,\frac{\sin\theta}{(a^2+r'^2-2ar'\cos\theta)^{1/2}} = 2\int_{-1}^{1} d\xi\,(a^2+r'^2+2ar'\xi)^{-1/2}$$

$$= \frac{2}{ar'}\left(a^2+r'^2+2ar'\xi\right)^{1/2}\Big|_{-1}^{1} = \frac{2}{ar'}\left(|a+r'|-|a-r'|\right) = \begin{cases}\dfrac{4}{a} & \text{for } r'\leq a\\[2ex] \dfrac{4}{r'} & \text{for } r'>a\end{cases} \qquad (3.5.4)$$

Next we double contract with $\hat{r}'\hat{r}'$ which gives

$$c_1+c_2 = \frac{1}{4\pi a^2}\int dS\,\frac{1}{|\vec{r}-\vec{r}\,'|}\left(1+\frac{(r'-r\cos\theta)^2}{|\vec{r}-\vec{r}\,'|^2}\right) = \frac{3c_1+c_2}{4} + \frac{1}{2}\int_{-1}^{1} d\xi\,\frac{(r'+a\xi)^2}{(a^2+r'^2+2ar'\xi)^{3/2}}$$

or

$$c_1+3c_2 = 2\int_{-1}^{1} d\xi\,\frac{(r'+a\xi)^2}{(a^2+r'^2+2ar'\xi)^{3/2}} = \frac{2}{3}\frac{a}{r'}\left[\frac{(\xi+2\,r'/a-a/r')^2-3\,(r'/a-a/r')^2}{(a^2+r'^2+2ar'\xi)^{1/2}}\right]_{-1}^{1}$$

$$= \begin{cases}4/3a & \text{for } r'\leq a\\[2ex] \dfrac{4}{3r'}\left(3-2\left(\dfrac{a}{r'}\right)^2\right) & \text{for } r'>a\end{cases} \qquad (3.5.5)$$

Combining eqs. (3.5.4) and (3.5.5) we find.

$$c_1(r') = \begin{cases}\dfrac{4}{3a} & \text{for } r'\leq a\\[2ex] \dfrac{1}{r'}\left(1+\dfrac{1}{3}\left(\dfrac{a}{r'}\right)^2\right) & \text{for } r'>a\end{cases} \qquad c_2(r') = \begin{cases}0 & \text{for } r'\leq a\\[2ex] \dfrac{1}{r'}\left(1-\left(\dfrac{a}{r'}\right)^2\right) & \text{for } r'>a\end{cases} \qquad (3.5.6)$$

Using these expressions eq. (3.5.3) becomes

$$\frac{1}{4\pi a^2}\int dS\,\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{1}}+\frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right) = \begin{cases}\dfrac{4}{3a}\,\vec{\vec{1}} & \text{for } r'\leq a\\[2ex] \dfrac{1}{r'}\left(\vec{\vec{1}}+\hat{r}'\hat{r}'\right)+\dfrac{a^2}{3r'^3}\left(\vec{\vec{1}}-3\hat{r}'\hat{r}'\right) & \text{for } r'>a\end{cases} \qquad (3.5.7)$$

In order to use this result we consider $\vec{G}_s$. The first and most important contribution is located on the surface of the sphere where $r'=a$. Of course this force on the surface also gives a velocity field which will reflect at the surface of the container and consequently leads to secondary contributions to $\vec{g}_s$ on the surface of the container. Faxén in fact proceeded to calculate these reflected velocity fields in a power serie in the size of the sphere divided by the distance to the surface of the container. As this depends on the shape of the container and is rather difficult we shall neglect these contributions. In that case averaging the velocity over the surface of the sphere, eq. (3.5.2), gives

$$\overline{\vec{v}_0(\vec{r})}^S \equiv \frac{1}{4\pi a^2}\int dS\, \vec{v}_0(\vec{r}) = -\frac{1}{6\pi\nu a}\int d\vec{r}\,'\, \vec{g}_s(\vec{r}\,') = -\frac{1}{6\pi\nu a}\vec{K} \qquad (3.5.8)$$

for the total force (per unit of mass) of the sphere on the fluid. The force of the fluid on the sphere is consequently

$$\vec{H} = 6\pi\eta a\, \overline{\vec{v}_0(\vec{r})}^S \qquad \underline{\text{Faxén's Theorem}} \qquad (3.5.9)$$

It is clear that if $\vec{v}_0(\vec{r}) = \vec{u}$ one obtains Stokes' result. Faxén's theorem is an extremely elegant and usefull result. It is true for a velocity field of arbitrary complexity (at low Reynolds numbers). It is also an example of a theorem which is easy to obtain using the general idea of induced forces without having to construct these forces explicitly. It is illuminating in this context to look at Faxén's original proof to see how difficult the analysis becomes if one does not use induced forces. If the sphere has itself a velocity $\vec{w}$ it follows, using Galilean invariance, that the force on the sphere is given by

$$\vec{H} = 6\pi\eta a\left(\overline{\vec{v}_0(\vec{r})}^S - \vec{w}\right) \qquad (3.5.10)$$

One may easily derive an alternative form of Faxén's theorem. For this we write

$$\overline{\vec{v}_0(\vec{r})}^S = \sum_{n=0}^{\infty}\frac{1}{n!}\,\overline{\vec{r}^{\,n}}^S \odot \left[\vec{\nabla}^n \vec{v}_0\right]_{\vec{r}=0} = \sum_{m=0}^{\infty}\frac{1}{(2m)!}\,\overline{\vec{r}^{\,2m}}^S \odot \left[\vec{\nabla}^{2m}\vec{v}_0\right]_{\vec{r}=0}$$

where $\odot$ is an n fold contraction. Clearly the n is odd terms average out to zero. The average $\overline{\vec{r}^{2m}}^{s}$ becomes a sum of products of m Kronecker deltas. Thus

$$\overline{\vec{v}_0(\vec{r})}^{s} = \sum_{m=0}^{\infty} \frac{a^{2m}}{(2m)!} c_m \left[\Delta^m \vec{v}_0\right]_{\vec{r}=0} \qquad (3.5.11)$$

where $c_0 = 1$, $c_1 = \frac{1}{3}$, ... We now use the fact that $\vec{v}_0$ has by definition no sources in the sphere. Thus for $r<a$ we have, using eqs. (3.2.1) and (3.1.6)

$$\Delta^2 \vec{v}_0 = \frac{1}{\eta} \operatorname{grad} \Delta P_0 = 0 \qquad \text{for } r<a \qquad (3.5.12)$$

Thus only the first two terms in eq. (3.5.11) contribute

$$\overline{\vec{v}_0(\vec{r})}^{s} = \vec{v}_0(0) + \frac{a^2}{6}\left[\Delta \vec{v}_0\right]_{\vec{r}=0} \qquad (3.5.13)$$

In this way one may write instead of eq. (3.5.9)

$$\vec{H} = 6\pi\eta a \left\{ \vec{v}_0(0) + \frac{a^2}{6}\left[\Delta \vec{v}_0\right]_{\vec{r}=0} \right\} \quad \underline{\text{Faxén's } 2^{nd} \text{ theorem}} \qquad (3.5.14)$$

If the functional form of $\vec{v}_0$ is known it is clearly easier to use this form of the theorem.

3.6 Hydrodynamic symmetry relations

In order to derive these symmetry relations we we write eq. (3.1.5) for the velocity in the following form

$$\frac{\partial}{\partial t}\vec{v} = -\frac{1}{\rho} \operatorname{div} \vec{\vec{P}} + \vec{G} \qquad , \qquad (3.6.1)$$

see also eqs (2.1.3) and (2.1.5) together with the definition of $\vec{G}$. We define the Fourier transform of the velocity field with respect to time as

$$\vec{v}(\vec{r},\omega) \equiv \int dt\, e^{i\omega t}\, \vec{v}(\vec{r},t) \Longleftrightarrow \vec{v}(\vec{r},t) = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\, \vec{v}(\vec{r},\omega) \qquad (3.6.2)$$

The Fourier transforms of all the other fields are defined in the same way. For these Fourier transformed fields eq. (3.6.1) becomes

$$-i\omega \vec{v} = -\frac{1}{\rho} \operatorname{div} \vec{\vec{P}} + \vec{G} \qquad (3.6.3)$$

Consider to sets of solutions $(\vec{v}_1, \vec{\vec{P}}_1, \vec{g}_1)$ and $(\vec{v}_2, \vec{\vec{P}}_2, \vec{g}_2)$ of this equation. Take a volume V in the system with a surface S and consider the following integral

$$\int_S \vec{n}\cdot\vec{\vec{P}}_1\cdot\vec{v}_2\, dS = \int_V \operatorname{div}(\vec{\vec{P}}_1\cdot\vec{v}_2)\, d\vec{r} = \int_V \left[(\operatorname{div}\vec{\vec{P}}_1)\cdot\vec{v}_2 + \vec{\vec{P}}_1 : \vec{\nabla}\vec{v}_2\right] d\vec{r}$$

Using eq. (3.6.3) and the fact that $\vec{v}_2$ is divergenless one may write this equation as

$$\int_S \vec{n}\cdot\vec{\vec{P}}_1\cdot\vec{v}_2\, dS = \int_V \left[i\omega\rho\,\vec{v}_1\cdot\vec{v}_2 + \rho\,\vec{g}_1\cdot\vec{v}_2 + \vec{\vec{\Pi}}_1 : \vec{\nabla}\vec{v}_2\right] d\vec{r}$$

$$= \int_V \left[i\omega\rho\,\vec{v}_1\cdot\vec{v}_2 + \rho\,\vec{g}_1\cdot\vec{v}_2 - 2\eta\,\vec{\vec{e}}_1 : \vec{\vec{e}}_2\right] d\vec{r} \qquad (3.6.4)$$

where $\vec{\vec{e}}_{1,2}$ are the rates of strain defined in eq. (1.2.2). Clearly the same relation is valid if one interchanges the two solutions. This gives the following <u>symmetry relation</u>

$$\int_S \vec{n}\cdot\left[\vec{\vec{P}}_1\cdot\vec{v}_2 - \vec{\vec{P}}_2\cdot\vec{v}_1\right] dS = \rho\int_V \left(\vec{g}_1\cdot\vec{v}_2 - \vec{g}_2\cdot\vec{v}_1\right) d\vec{r} \qquad (3.6.5)$$

In order to do the next step we choose a special solution for $\vec{v}_2$. Take a Stokeslet in $\vec{r}=0$ (the choice of the location is only done in order to simplify the formulea)

$$\vec{v}_2(\vec{r}) = \frac{1}{r}\,\vec{\alpha}\cdot(\vec{\vec{1}} + \hat{r}\hat{r}) \qquad \vec{g}_2(\vec{r}) = 8\pi\nu\,\vec{\alpha}\,\delta(\vec{r})$$

$$\left.\begin{aligned} p_2(\vec{r}) &= 2\eta\,\frac{1}{r^2}\,\vec{\alpha}\cdot\hat{r} \\ \vec{\vec{\Pi}}_2(\vec{r}) &= -2\eta\,\frac{1}{r^2}(\vec{\alpha}\cdot\hat{r})(\vec{\vec{1}} - 3\hat{r}\hat{r}) \end{aligned}\right\} \vec{\vec{P}}_2(\vec{r}) = 6\eta\,\frac{1}{r^2}(\vec{\alpha}\cdot\hat{r})\,\hat{r}\hat{r} \qquad (3.6.6)$$

Consider a steady state solution $(\vec{v}(\vec{r}), \vec{\vec{P}}(\vec{r}), \vec{g}(\vec{r}))$ for which the sources are all outside V, i.e. $\vec{g}=0$ inside V. Choose the center of the coordinate frame in an arbitrary point in V. Then eq. (3.6.5) together with (3.6.6) gives

$$\vec{\alpha}\cdot\vec{v}(0) = \frac{1}{8\pi\eta}\int_S \left[\frac{6\eta}{r^2}(\vec{n}\cdot\hat{r})(\vec{\alpha}\cdot\hat{r})(\hat{r}\cdot\vec{v}) - \frac{1}{r}\,\vec{n}\cdot\vec{\vec{P}}\cdot(1+\hat{r}\hat{r})\cdot\vec{\alpha}\right] dS \qquad (3.6.7)$$

And in view of the fact that this is true for all choices of $\vec{\alpha}$ we have

$$\vec{v}(0) = \frac{1}{8\pi\eta}\int_S \left[\frac{6\eta}{r^2}\left(\vec{n}\cdot\hat{r}\hat{r}\hat{r}\cdot\vec{v}\right)\hat{r} - \frac{1}{2}\vec{n}\cdot\vec{\vec{P}}\cdot\left(1+\hat{r}\hat{r}\right)\right] dS \qquad (3.6.8)$$

It follows from this identity that: once the velocity and the gradient of the velocity are known on the surface of a source free region the velocity field inside the region is completely determined. One may generalise the above result to non steady flow which corresponds to finite frequencies. This makes it necessary to calculate the velocity field due to an oscillating point force which we have not done yet. Identities, like the one given above, are well known in the theory of differential equations.

3.7 Flow due to a spherically symmetric force distribution

Consider a force distribution which is time independent and spherically symmetric.

$$\vec{G} = \vec{G}(r) \qquad (3.7.1)$$

According to eq. (3.2.12) the velocity field is given by

$$\vec{v}(\vec{r}) = \frac{1}{8\pi\nu}\int d\vec{r}\,'\, \vec{G}(r')\cdot\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{1}} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right)\right]$$

$$= \frac{1}{2\nu}\int_0^\infty dr'\, r'^2\, \vec{G}(r')\left\{\frac{1}{4\pi}\int d\Omega\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\left(\vec{\vec{1}} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right)\right]\right\} \qquad (3.7.2)$$

The expression between the curly brackets is simply the average of the expression between square brackets over a sphere with radius r'. This average we have in fact calculated and is given in eq. (3.5.7) where we must replace a by r' and r' by r. Substitution in eq. (3.7.2) gives

$$\vec{v}(\vec{r}) = \frac{2}{3\nu}\int_r^\infty dr'\, r'\,\vec{G}(r') + \frac{1}{2\nu}\left[\frac{1}{r}\left(\vec{\vec{1}}+\hat{r}\hat{r}\right)\right]\int_0^r dr'\, r'^2\,\vec{G}(r')$$

$$+ \frac{1}{6\nu}\left[\frac{1}{r^3}\left(\vec{\vec{1}}-3\hat{r}\hat{r}\right)\right]\int_0^r dr'\, r'^4\,\vec{G}(r') \qquad (3.7.3)$$

Thus the velocity field has a Stokeslet and a potential doublet like term of which the strength is given in terms of the force distribution for $r' < r$

For the Stokes problem we had (see eq. (3.4.9))

$$\vec{G} = -\frac{3\nu\vec{u}}{2a}\,\delta(r-a) \qquad (3.7.4$$

This is clearly a spherically symmetric force distribution. As is easy to verify one obtains the Stokes field upon substitution in eq. (3.7.3) if one adds the homogeneous field $\vec{u}$ which is generated in infinity.

If the force distribution is restricted to a finite region the velocity field <u>outside</u> <u>this</u> <u>region</u> is the sum of the Stokeslet and potential doublet field with

$$\vec{\alpha} = \frac{1}{2\nu}\int_0^\infty dr'\, r'^2\, \vec{G}(r') \quad \text{and} \quad \vec{\beta} = \frac{1}{12\nu}\int_0^\infty dr'\, r'^4\, \vec{G}(r') \qquad (3.7.5$$

This are exactly the expressions which follow in the context of the multipole expansion discussed in paragraph (3.3). Because of symmetry considerations only these two multipoles contribute to the field outside the force distribution in the present case.

<u>3.8 The force on a solid body moving in a fluid</u>

Consider a solid body of arbitrary shape moving with a constant velocity $\vec{w}$ in a fluid. The fluid motio[n] is due to the motion of this body alone. As the solid body may simply be considered as a region of the flui[d] where the mass density is different and where the force density is such that the region stays rigid one may use the Navier-Stokes equation also in this region. Integrating the Navier-Stokes equation over the volume of the solid body one obtains

$$m\,\frac{\partial\vec{w}}{\partial t} = -\int_V \operatorname{div}\vec{\vec{P}}\,d\vec{r} + \int_V \rho\vec{F}\,d\vec{r} = -\int_S \vec{n}\cdot\vec{\vec{P}}\,dS + \int_V \rho\,\vec{G}\,d\vec{r}$$

$$= -\int_S \vec{n}\cdot\vec{\vec{P}}\,dS + \vec{H} \qquad (3.8.1)$$

$\vec{H}$ is the integral over the force density in the body and is the result of external forces "pushing" the body, internal forces which keep the body rigid and the forces on the surface of the body acting on the fluid. The sum of these forces $\vec{H}$ is clearly the sum of the external forces pushing the body through the fluid. The integral of $\vec{n}\cdot\vec{\vec{P}}$ over the surface of the body gives the total force of the fluid on the solid body. If the velocity of the body $\vec{w}$ is constan[t] this frictional force $\vec{H}_{fric}$ balances the other force $\vec{H}$

$$\vec{H}_{fric} = -\int_S \vec{n}\cdot\vec{\vec{P}}\,dS = -\vec{H} \qquad (3.8.2)$$

For low Reynolds numbers there is a linear relation between the frictional force and the velocity of the body

$$\vec{H}_{fric} = -\vec{\vec{\zeta}}\cdot\vec{W} \qquad (3.8.3)$$

$\vec{\vec{\zeta}}$ is the friction tensor. For a sphere with radius a Stoke[s] gives as friction tensor

$$\vec{\vec{\zeta}} = 6\pi\eta a \vec{\vec{1}} \qquad (3.8.4)$$

For a body of general shape the friction tensor is not a constant times the unit tensor. One may easily show that the tensor is symmetric

$$\zeta_{ij} = \zeta_{ji} \qquad (3.8.5)$$

In order to proof this we consider two sets of solutions $(\vec{W}_1, \vec{V}_1, \vec{\vec{P}}_1, \vec{H}_{fr,1})$ and $(\vec{W}_2, \vec{V}_2, \vec{\vec{P}}_2, \vec{H}_{fr,2})$ Using the fact that the force densities $\vec{g}_{1,2}$ are zero outside the body it follows from the hydrodynamic symmetry relation (3.6.5) that

$$0 = \int_S \vec{n}\cdot[\vec{\vec{P}}_1\cdot\vec{V}_2 - \vec{\vec{P}}_2\cdot\vec{V}_1]\,dS = \int_S \vec{n}\cdot[\vec{\vec{P}}_1\cdot\vec{W}_2 - \vec{\vec{P}}_2\cdot\vec{W}_1]\,dS = -\vec{H}_{fr,1}\cdot\vec{W}_2 + \vec{H}_{fr,2}\cdot\vec{W}_1$$

Substituting eq. (3.8.3) one obtains

$$\vec{W}_2\cdot\vec{\vec{\zeta}}\cdot\vec{W}_1 = \vec{W}_1\cdot\vec{\vec{\zeta}}\cdot\vec{W}_2 \qquad (3.8.6)$$

As this is true for all possible choices of $\vec{W}_1$ and $\vec{W}_2$ the friction tensor is clearly symmetric. This symmetry relation is an example of an Onsager relation.

Elementary incoming solutions in incompressible fluid

a $\vec{v}(\vec{r}) = \vec{u}$ $\qquad p(\vec{r}) = 0$ $\qquad$ vectorial source

b $\vec{v}(\vec{r}) = A r^2 (2 - \hat{r}\hat{r}) \cdot \vec{u}$ $\qquad p(\vec{r}) = 10 \eta A \, \vec{r} \cdot \vec{u}$

a $\vec{v}(\vec{r}) = \overleftrightarrow{\alpha} \cdot \vec{r}$ $\qquad p(\vec{r}) = 0$ $\qquad$ $\overleftrightarrow{\alpha}$ symm. traceless

b $\vec{v}(\vec{r}) = B r^3 \overleftrightarrow{\alpha} : \hat{r} (\hat{r}\hat{r} - \frac{5}{2} \overleftrightarrow{I})$ $\qquad p(\vec{r}) = -\frac{21}{2} B \eta r^2 \overleftrightarrow{\alpha} : \hat{r}\hat{r}$

3.9 The general solution

Most of this section we considered only steady flow. In this paragraph we shall construct the solution for the general case of non-steady flow. The equations to be solved are, cf. eqs. (3.1.5) and (3.1.6),

$$\frac{\partial}{\partial t}\vec{v} - \nu\Delta\vec{v} = -\vec{\nabla}\frac{p}{\rho} + \vec{g} \qquad (3.9.1)$$

$$\Delta\left(\frac{p}{\rho}\right) = \vec{\nabla}\cdot\vec{g} \qquad (3.9.2)$$

In order to construct the solution we use the Fourier-transformed fields. The definition for for instance the velocity field is given by

$$\vec{v}(\vec{k},\omega) \equiv \int d\vec{r}\,dt\; e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,\vec{v}(\vec{r},t) \qquad (3.9.3)$$

The inverse transformation is

$$\vec{v}(\vec{r},t) = \frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\vec{v}(\vec{k},\omega) \qquad (3.9.4)$$

The Fourier transforms of p and $\vec{g}$ are defined in the same way.

Fourier transformation of eqs. (3.9.1) and (3.9.2) gives

$$-i\omega\,\vec{v}(\vec{k},\omega) + \nu k^2\,\vec{v}(\vec{k},\omega) = -i\vec{k}\,\frac{p(\vec{k},\omega)}{\rho} + \vec{g}(\vec{k},\omega) \qquad (3.9.5)$$

$$-k^2\,\frac{p(\vec{k},\omega)}{\rho} = i\vec{k}\cdot\vec{g}(\vec{k},\omega) \qquad (3.9.6)$$

That these equations result is most easily seen if one verifies using eq. (3.9.4)

$$F.T.\left(\frac{\partial}{\partial t}\right) = -i\omega \quad \text{and} \quad F.T.(\vec{\nabla}) = i\vec{k} \qquad (3.9.7)$$

The Fourier transformed equations are easy to solve. In fact one finds for the pressure from eq. (3.9.6)

$$p(\vec{k},\omega) = -\rho\,k^{-2}\,i\vec{k}\cdot\vec{g}(\vec{k},\omega) \qquad (3.9.8)$$

Substituting this in eq. (3.9.5) one finds for the velocity

$$\vec{v}(\vec{k},\omega) = (-i\omega + \nu k^2)^{-1} (\vec{\vec{I}} - \hat{k}\hat{k}) \cdot \vec{G}(\vec{k},\omega) \tag{3.9.9}$$

We now define the following propagator for time dependent fluid flow in an incompressible fluid

$$\vec{\vec{T}}(\vec{k},\omega) \equiv 8\pi\nu (-i\omega + \nu k^2)^{-1} (\vec{\vec{I}} - \hat{k}\hat{k}) \tag{3.9.10}$$

The general solution may then be written as

$$\vec{v}(\vec{k},\omega) = \vec{\vec{T}}(\vec{k},\omega) \cdot \frac{1}{8\pi\nu} \vec{G}(\vec{k},\omega) \tag{3.9.11}$$

As it is clearly interesting to see the spacial dependence of this solution more clearly we will transform eq. (3.9.11) back to functions of position and frequency

$$\vec{v}(\vec{r},\omega) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{\vec{T}}(\vec{k},\omega) \cdot \frac{1}{8\pi\nu} \vec{G}(\vec{k},\omega)$$

$$= \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{\vec{T}}(\vec{k},\omega) \cdot \int d\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} \frac{1}{8\pi\nu} \vec{G}(\vec{r}\,',\omega)$$

$$= \int d\vec{r}\,' \left[\frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, \vec{\vec{T}}(\vec{k},\omega) \right] \cdot \frac{1}{8\pi\nu} \vec{G}(\vec{r}\,',\omega)$$

This leads to the following expression

$$\vec{v}(\vec{r},\omega) = \int d\vec{r}\,'\, \vec{\vec{T}}(\vec{r}-\vec{r}\,',\omega) \cdot \frac{1}{8\pi\nu} \vec{G}(\vec{r}\,',\omega) \tag{3.9.12}$$

The inverse Fourier transform of the propagator is

$$\vec{\vec{T}}(\vec{r},\omega) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{\vec{T}}(\vec{k},\omega)$$

$$= \frac{e^{-\alpha r}}{r} (\vec{\vec{I}} + \hat{r}\hat{r}) + 2\alpha^{-2} \left[(1 + \alpha r + \tfrac{1}{2}\alpha^2 r^2) e^{-\alpha r} - 1 \right] \left(\frac{\vec{\vec{I}} - 3\hat{r}\hat{r}}{r^3} \right) \tag{3.9.13}$$

Here α is the socalled inverse screening length

$$\alpha \equiv \sqrt{\frac{-i\omega}{\nu}} \qquad \text{with } \operatorname{Re}\alpha > 0 \tag{3.9.14}$$

To show that one obtains eq. (3.9.13) if one substitutes eq. (3.9.10) is rather laborious and we will therefore

The potential doublet field due to the source

$$\vec{g}(\vec{r}) = 8\pi\nu\vec{\beta}\,\Delta\,\delta(\vec{r})$$

is obtained by evaluating $\Delta\vec{V}$ in eq. (3.9.16)
One obtains

$$\vec{V}(\vec{r},\omega) = 2\,\frac{e^{-\alpha r}}{r^3}\left(1+\alpha r+\tfrac{1}{2}\alpha^2 r^2\right)\left(\vec{\vec{1}}-3\hat{r}\hat{r}\right)\cdot\vec{\beta} + 2\alpha^2\,\frac{e^{-\alpha r}}{r}\left(\vec{\vec{1}}+\hat{r}\hat{r}\right)\cdot\vec{\beta}$$

This again reduces to the result given in eq. (3.3.29) for zero frequency.

not do this explicitly. In order to understand the role of the propagator better we calculate again the field due to a point force

$$\vec{G}(\vec{r}) = 8\pi\nu\, \vec{u}\, \delta(\vec{r}) \tag{3.9.15}$$

Substitution in eq. (3.9.12) gives

$$\vec{v}(\vec{r},\omega) = \int d\vec{r}'\, \overset{\Rightarrow}{T}(\vec{r}-\vec{r}',\omega)\cdot\vec{u}\,\delta(\vec{r}') = \overset{\Rightarrow}{T}(\vec{r},\omega)\cdot\vec{u}$$

$$= \frac{e^{-\alpha r}}{r}(\overset{\Rightarrow}{I}+\hat{r}\hat{r})\cdot\vec{u} + 2\alpha^{-2}\left[(1+\alpha r+\tfrac{1}{2}\alpha^2 r^2)e^{-\alpha r}-1\right]\left(\frac{\overset{\Rightarrow}{I}-3\hat{r}\hat{r}}{r^3}\right)\cdot\vec{u} \tag{3.9.16}$$

The velocity field due to a point source is thus given by the propagator. Also we see that for finite frequencies the velocity field due to a point source is a combination of a Stokeslet and a potential doublet like field. In principle it would be most proper to call the above field a frequency dependent Stokeslet. For zero frequency one has

$$\overset{\Rightarrow}{T}(\vec{r},\omega=0) = \frac{1}{r}(\overset{\Rightarrow}{I}+\hat{r}\hat{r}) \tag{3.9.17}$$

The corresponding velocity field then reduces to the Stokeslet field given in eq. (3.2.11).

The zeros of the denominator $(-i\omega+\nu k^2)$ of the propagator correspond to socalled free solutions, where free indicates that there is no force density. In the present case the free solution is

$$\vec{v}(\vec{r},t) = \vec{u}\,\exp\left[i\vec{k}\cdot\vec{r}-i\omega(k)t\right]$$

$$= \vec{u}\,\exp\left[i\vec{k}\cdot\vec{r}-\nu k^2 t\right] \tag{3.9.18}$$

where $\vec{u}\cdot\vec{k}=0$ to make the solution divergenceless.

$$\omega(k) = -i\nu k^2 \tag{3.9.19}$$

is the dispersion relation for these solutions. The free solutions are damped. A more interesting free solution exists in the compressible case corresponding to sound.

List of Fourier transforms

1	$\delta(\vec{r})$
$i\vec{k}$	$\vec{\nabla}\,\delta(\vec{r})$
$(i\vec{k})^n$	$(\vec{\nabla})^n\,\delta(\vec{r})$
$-k^2$	$\Delta\ \delta(\vec{r})$
$(-k^2)^m$	$\Delta^m\,\delta(\vec{r})$

Powers of $\vec{k}$ correspond to gradients of $\delta(\vec{r})$

$(i\vec{k})^n f(\vec{k})$	$(\vec{\nabla})^n f(\vec{r})$
$(-k^2)^m f(\vec{k})$	$\Delta^m f(\vec{r})$

Powers of $\vec{k}$ times a function of $\vec{k}$ correspond to gradients of the inverse fourier transform of this function

$\frac{1}{k^2}$	$\frac{1}{4\pi r}$	Coulomb potential
$\hat{k}\hat{k} = \frac{\vec{k}\vec{k}}{k^2}$	$\frac{1}{3}\vec{\vec{1}}\delta(\vec{r}) - \frac{1}{4\pi r^3}(\vec{\vec{1}} - 3\hat{r}\hat{r}) = -\vec{\nabla}\vec{\nabla}\frac{1}{4\pi r}$	
$\frac{\hat{k}\hat{k}}{k^2} = \frac{\vec{k}\vec{k}}{k^4}$	$\frac{1}{8\pi r}(\vec{\vec{1}} - \hat{r}\hat{r})$	
$\vec{\vec{T}}(\vec{k}, w=0) = \frac{\vec{\vec{1}} - \hat{k}\hat{k}}{k^2}$	$\frac{1}{8\pi r}(\vec{\vec{1}} + \hat{r}\hat{r})$	gives Stokeslet velocity field

The stationary velocity fields due to higher order force multipoles are given by

$$(i\vec{k})^n\,\vec{\vec{T}}(\vec{k}, w=0) = (i\vec{k})^n\,\frac{\vec{\vec{1}} - \hat{k}\hat{k}}{k^2} \quad \Big| \quad (\vec{\nabla})^n\,\frac{1}{8\pi r}(\vec{\vec{1}} + \hat{r}\hat{r})$$

The potential doublet is given by

$$-k^2\,\vec{\vec{T}}(\vec{k}, w=0) = \vec{\vec{1}} - \hat{k}\hat{k} \quad \Big| \quad \frac{2}{3}\vec{\vec{1}}\,\delta(\vec{r}) + \frac{1}{4\pi r^3}(\vec{\vec{1}} - 3\hat{r}\hat{r})$$

The stationary potential flow with the same symmetry as the force multipole fields are

$(i\vec{k})^n\left(-k^2\overleftrightarrow{T}(\vec{k},\omega=0)\right)=(i\vec{k})^n\left(\overleftrightarrow{1}-\hat{k}\hat{k}\right)$	$(\vec{\nabla})^n\left[\frac{2}{3}\overleftrightarrow{1}\delta(\vec{r})+\frac{1}{4\pi r^3}\left(\overleftrightarrow{1}-3\hat{r}\hat{r}\right)\right]$

If one applies another $(-k^2)$ to the potential doublet one obtains $k^2-\vec{k}\vec{k}$. As this gives only derivatives of $\delta(\vec{r})$ there are no other velocity fields with the same symmetry. Applying $(-k^2)$ again only gives higher order derivatives of $\delta(\vec{r})$

$\dfrac{1}{k^2+\alpha^2}$	$\dfrac{1}{4\pi r}e^{-\alpha r}$
$\dfrac{\vec{k}\vec{k}}{k^2+\alpha^2}$	$\frac{1}{3}\overleftrightarrow{1}\delta(\vec{r})-\frac{1}{4\pi r^3}\left(\overleftrightarrow{1}-3\hat{r}\hat{r}\right)e^{-\alpha r}\left(1+\alpha r+\frac{1}{4}\alpha^2r^2\right)+\frac{\alpha^2}{16\pi r}\left(\overleftrightarrow{1}+\hat{r}\hat{r}\right)e^{-\alpha r}$
$\dfrac{\hat{k}\hat{k}}{k^2+\alpha^2}$	$\frac{1}{16\pi r}\left(\overleftrightarrow{1}+\hat{r}\hat{r}\right)e^{-\alpha r}-\frac{1}{4\pi r^3\alpha^2}\left(\overleftrightarrow{1}-3\hat{r}\hat{r}\right)\left[e^{-\alpha r}\left(1+\alpha r+\frac{1}{4}\alpha^2r^2\right)-1\right]$
$\overleftrightarrow{T}(\vec{k},\omega)=\dfrac{\overleftrightarrow{1}-\hat{k}\hat{k}}{k^2+\alpha^2}$	$\frac{1}{8\pi r}\left(\overleftrightarrow{1}+\hat{r}\hat{r}\right)e^{-\alpha r}-\frac{1}{4\pi r^3\alpha^2}\left(\overleftrightarrow{1}-3\hat{r}\hat{r}\right)\left[e^{-\alpha r}\left(1+\alpha r+\frac{1}{2}\alpha^2r^2\right)-1\right]$

Here $\alpha\equiv\sqrt{\frac{-i\omega}{\nu}}$ with $\mathrm{Re}\,\alpha\geqslant 0$. The frequency dependent velocity fields due to higher order force multipoles are again found by applying $(i\vec{k})^n$ to the left hand side and $(\vec{\nabla})^n$ to the right hand side. The potential doublet is therefore given by

$$-k^2\overleftrightarrow{T}(\vec{k},\omega)=\alpha^2\overleftrightarrow{T}(\vec{k},\omega)-(-k^2)\overleftrightarrow{T}(\vec{k},\omega=0)=\alpha^2\overleftrightarrow{T}(\vec{k},\omega)-\left(\overleftrightarrow{1}-\hat{k}\hat{k}\right)$$

Applying $(-k^2)$ again gives

$$(-k^2)^2\overleftrightarrow{T}(\vec{k},\omega)=\alpha^2\left(-k^2\overleftrightarrow{T}(\vec{k},\omega)\right)+\left(k^2-\vec{k}\vec{k}\right)$$

$$(-k^2)^n\overleftrightarrow{T}(\vec{k},\omega)=\alpha^{2(n-1)}\left(-k^2\overleftrightarrow{T}(\vec{k},\omega)\right)+\left(\text{polynomial in } k^2\right)\left(k^2-\vec{k}\vec{k}\right)$$

Neglecting the δ function like contributions the general solution of a spherically symmetric source is thus given by

$$\left[A+\alpha^2 f(\alpha^2)\right]\overleftrightarrow{T}(\vec{k},\omega)-f(\alpha^2)\left(\overleftrightarrow{1}-\hat{k}\hat{k}\right)$$

where A is the total force strength and $f(\alpha^2)$ an appropriately chosen function of the frequency. For higher order multipoles one simply applies $(i\vec{k})$ to the appropriate power. The $\vec{r}$ dependence can be found using the above formulae.

4 Application to suspensions

4.1 Introduction

A suspension is a fluid in which small elements of an other material are suspended. A typical example is fog or a cloud in which small water droplets are suspended in air. An other example is a river in which small solid pieces of rock are suspended (a muddy river). In chemistry one has suspensions of polymers In fact colloidal chemistry is a field in which one is in general interested in fluids in which another substance is suspended. In all these systems the size of the suspended particles is small in the order of a micron or less. In the gravitational field the velocity of such a particle is of the order of

$$|\vec{w}| \sim (\rho_s-\rho)\,g\,\frac{\frac{4\pi}{3}a^3}{6\pi\eta a} = \frac{2}{9}(\rho_s-\rho)\,g\,\frac{a^2}{\eta}$$

Using $(\rho_s-\rho)\sim 1$, $g \simeq 1000$, $a = 10^{-4}$, $\eta = 0.02$ all in c.g.s. units one obtains

$$|\vec{w}| \sim 10^{-4}\ \text{cm/sec}$$

If one studies the flow of a suspension one may thus for all practical purposes assume that the suspended particles move with the fluid. Only in special cases the relative motion is a matter of interest. If one wants to study sedimentation for instance it is precisely this relative motion which is being considered.

In our further analysis we will assume the suspended particles to be spherical. This is of course a simplification. As we have discussed, however, one may use a multipole expansion of the force distribution on the surface of the particle. The Stokeslet strength is then the most important source of a velocity field and the stresslet and the rotlet are the next most important sources etc. It is clear that only close to the particles the shape is really important, somewhat further away the shape becomes unimportant. Because of this the assumption of spherical particles is not quite as restrictive as it may seem.

An important aspect of the behaviours of a suspension is the hydrodynamic interaction between the suspended particles. All the suspended particles affect each others motion through the velocity field caused by the particles. Because of this correlations in the positio will be important for the flow. On the other hand the flow will also lead to correlations in the positions of the particles. A subject which is presently of great interest is two-phase flow. This general field, in whic many different flow types are observed, is rather difficult to describe in detail. Suspensions are a relatively simple variety of two-phase flow for which many properties may be calculated in more detail.

4.2 Sedimentation

In a sedimentation experiment one measures the rate at which suspended particles with a somewhat larger specific weight than the specific weight of the fluid will sediment on the bottom of the container. The veloci ties are usually extremely small so that the Reynolds number is small. This is even true in a centrifuge if one considers a frame of reference which rotates with the centrifuge. In such an experiment the forces are known and one wants to calculate the velocities of the particles. As the system is linear for low Reynolds numbers one may write

$$\vec{W}_n = -\sum_{m=1}^{N} \overset{\Rightarrow}{\mu}_{nm} \cdot \vec{H}_m \qquad (4.2.1$$

where $(-\vec{H}_m)$ is the gravitational force on the m^{th} particle. Because of the small velocities and accelerations involved one may neglect inertial effects, i.e. $d\vec{W}_n/dt$, so that the gravitational force on the particles is equal to minus the hydrodynamic frictional force $\vec{H}_n$. The coefficients $\overset{\Rightarrow}{\mu}_{nm}$ are called <u>mobility tensors</u> or <u>mobilities</u>. One may alternatively express the forces in the velocities

$$\vec{H}_n = -\sum_{m=1}^{N} \overset{\Rightarrow}{\xi}_{nm} \cdot \vec{W}_m \qquad (4.2.2$$

The <u>friction tensors</u> are related to the mobilities by

$$\sum_{m=1}^{N} \overleftrightarrow{\mu}_{nm} \cdot \overleftrightarrow{\zeta}_{mk} = \delta_{nk} \overleftrightarrow{1} \qquad (4.2.3)$$

In the sedimentation experiment the velocity in the absence of the spheres would be zero. The fluid velocity field is caused by the motion of the spheres. Also in the fluid one may neglect inertial effects and we may thus use the general solution for the steady state case given in eq. (3.2.12)

$$\vec{v}(\vec{r}) = \frac{1}{8\pi\nu} \sum_{m=1}^{N} \int d\vec{r}\,' \vec{g}_m(\vec{r}\,') \cdot \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(\overleftrightarrow{1} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right)\right] \qquad (4.2.4)$$

In this equation $\vec{g}_m$ is the force density on the fluid exerted by sphere m. The forces on the surface of the container are neglected. Averaging this equation over the surface of the n^{th} sphere one has

$$\vec{W}_n = \overline{\vec{v}(\vec{r})}^{S_n} = \frac{1}{8\pi\nu} \sum_{m=1}^{N} \int d\vec{r}\,' \vec{g}_m(\vec{r}\,') \cdot \overline{\left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(\overleftrightarrow{1} + \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right)\right]}^{S_n}$$

Using eq. (3.5.7) for the average one obtains, a_n is the radius and $\vec{R}_n$ the position of sphere

$$\vec{W}_n = -\frac{1}{6\pi\eta a_n} \vec{H}_n + \frac{1}{8\pi\nu} \sum_{m\neq n} \int d\vec{r}\,' \vec{g}_m(\vec{r}\,') \left\{\frac{1}{|\vec{r}\,'-\vec{R}_n|}\left(\overleftrightarrow{1} + \frac{(\vec{r}\,'-\vec{R}_n)(\vec{r}\,'-\vec{R}_n)}{|\vec{r}\,'-\vec{R}_n|^2}\right) + \frac{a_n^2}{3|\vec{r}\,'-\vec{R}_n|^3}\left(\overleftrightarrow{1} - 3\frac{(\vec{r}\,'-\vec{R}_n)(\vec{r}\,'-\vec{R}_n)}{|\vec{r}\,'-\vec{R}_n|^2}\right)\right\} \qquad (4.2.5)$$

In order to proceed beyond this formula one must introduce approximations. The usual procedure is to take the radii of the particles small compared to the distance between the particles. This corresponds to using again a multipole expansion. We shall restrict ourselves to the lowest order multipole which corresponds to replacing $\vec{r}\,'$ by $\vec{R}_m$ between the curly brackets in eq. (4.2.5) and neglecting the term proportional to a_n^2. The equation then reduces to

$$\vec{W}_n = -\frac{1}{6\pi\eta a_n} \vec{H}_n - \frac{1}{8\pi\eta} \sum_{m\neq n} \frac{1}{R_{nm}} \left(\overleftrightarrow{1} + \hat{R}_{nm} \hat{R}_{nm}\right) \cdot \vec{H}_m \qquad (4.2.6)$$

where $\vec{R}_{nm} \equiv \vec{R}_n - \vec{R}_m$, $R_{nm} \equiv |\vec{R}_{nm}|$ and $\hat{R}_{nm} \equiv \vec{R}_{nm} / R_{nm}$

The resulting mobilities are

$$\overset{\Rightarrow}{\mu}_{nm} = \begin{cases} \dfrac{1}{6\pi\eta a_n}\overset{\Rightarrow}{I} & \text{for } n=m \\ \dfrac{1}{8\pi\eta R_{nm}}\left(\overset{\Rightarrow}{I}+\hat{R}_{nm}\hat{R}_{nm}\right) & \text{for } n\neq m \end{cases} \qquad (4.2.7)$$

If, as in the case of the gravitational force for spheres with equal diameters, all forces on the particles are equal, $\vec{F}_m = -\vec{g}$, one obtains as velocity of the n^{th} particle

$$\vec{w}_n = \frac{1}{6\pi\eta a_n}\left\{1+\frac{3}{4}\sum_{m\neq n}\frac{a_n}{R_{nm}}\left(\overset{\Rightarrow}{I}+\hat{R}_{nm}\hat{R}_{nm}\right)\right\}\cdot\vec{g} \qquad (4.2.8)$$

It is clear that due to the presence of the other spheres the velocity of the particles increases. This effect is most pronounced when the spheres are close together

Using the fact that the friction tensors are, so to say the inverse of $\overset{\Rightarrow}{\mu}_{nm}$ one may show that

$$\overset{\Rightarrow}{\zeta}_{nn} = 6\pi\eta a_n\left\{\overset{\Rightarrow}{I}+\left(\frac{3}{4}\right)^2\sum_{m\neq n}\frac{a_n a_m}{R_{nm}^2}\left(\overset{\Rightarrow}{I}+\hat{R}_{nm}\hat{R}_{nm}\right)\cdot\left(\overset{\Rightarrow}{I}+\hat{R}_{mn}\hat{R}_{mn}\right)\right\}$$

$$= 6\pi\eta a_n\left\{\overset{\Rightarrow}{I}+\left(\frac{3}{4}\right)^2\sum_{m\neq n}\frac{a_n a_m}{R_{nm}^2}\left(\overset{\Rightarrow}{I}+3\,\hat{R}_{nm}\hat{R}_{nm}\right)\right\}$$

$$\overset{\Rightarrow}{\zeta}_{nm} = -6\pi\eta a_n a_m\left\{\frac{3}{4R_{nm}}\left(\overset{\Rightarrow}{I}+\hat{R}_{nm}\hat{R}_{nm}\right)-\left(\frac{3}{4}\right)^2\sum_{k\neq n,m}\frac{a_k}{R_{nk}R_{km}}\left(\overset{\Rightarrow}{I}+\hat{R}_{nk}\hat{R}_{nk}\right)\cdot\left(\overset{\Rightarrow}{I}+\hat{R}_{km}\hat{R}_{km}\right)\right\}$$

$$\text{for } n\neq m \qquad (4.2.9)$$

Up to second order in one over the distance between the particles. The simplest way to check this result is to substitute $\overset{\Rightarrow}{\zeta}_{mk}$ and $\overset{\Rightarrow}{\mu}_{nm}$ into eq. (4.2.3) and to verify that, if all terms which go to zero faster than one over the distance squared are neglected, the equation is valid. It is easy however, to understand the source of the one over the distance squared terms: particle m has a velocity $\vec{w}_m$ this results in a Stokeslet field around particle m; particle k feels this velocity field and as a consequence the force on that particle is changed from $-6\pi\eta a_k\,\vec{w}_k$ to $-6\pi\eta a_k(\vec{w}_k - \text{Stokeslet field})$; the Stokeslet field from particle k is thus modified and particle n feels this modified field. This gives the second term in $\overset{\Rightarrow}{\zeta}_{nm}$. This last method to find eq. (4.2.9) is called the method of multiple reflections.

In the calculation of the mobilities we have used the monopole approximation where the total force on the spheres is located in the center of the spheres. A somewhat similar but nevertheless somewhat nicer approximation is to spread the total force out over the surface of the sphere

$$\vec{g}_m(\vec{r}\,') = \frac{-1}{4\pi a_m^2 \varsigma} \vec{H}_m \, \delta(r-a) \qquad (4.2.10)$$

In this approximation one also takes the potential doublet velocity field along and as a consequence the boundary condition on the surface of the spheres is satisfied for the average velocity $\vec{w}_n$ whereas this is not the case if one neglects the case if the potential doublet is neglected. Note in this respect that if one neglects velocity fields due to higher order multipoles that the field close to the sphere is no longer the exact field. As a consequence the approximated field no longer satisfies the boundary condition. Substituting eq. (4.2.10) in eq. (4.2.5) one obtains

$$\vec{w}_n = -\frac{1}{6\pi\eta a_n}\vec{H}_n - \frac{1}{8\pi\eta}\sum_{m\neq n}\left\{\frac{1}{R_{nm}}\left(\vec{\vec{1}} + \hat{R}_{nm}\hat{R}_{nm}\right) + \frac{a_n^2 + a_m^2}{3R_{nm}^3}\left(\vec{\vec{1}} - 3\hat{R}_{nm}\hat{R}_{nm}\right)\right\}\cdot\vec{H}_m \qquad (4.2.11)$$

The tensor between curly brackets is the so called Rotny-Prager tensor. The mobilities and the friction tensors are if one uses eq. (4.2.11) instead of eq. (4.2.6) accordingly modified. As one may easily give these somewhat more complicated expressions we shall not give them explicitly. The corrections are obviously of higher than the second order in the radius over the distance.

4.3 Diffusion

Due to thermal motion the suspended spheres diffuse through the fluid. The probability distribution for the spheres satisfies the following equation of motion

$$\frac{\partial}{\partial t} P(\vec{w}_1, \cdots, \vec{w}_N, \vec{R}_1, \cdots, \vec{R}_N, t)$$
$$= - \sum_{n=1}^{N} \left(\frac{1}{m_n} \frac{\partial}{\partial \vec{w}_n} \cdot \vec{F}_n + \frac{\partial}{\partial \vec{R}_n} \cdot \vec{w}_n \right) P(\vec{w}_1, \cdots, \vec{w}_N, \vec{R}_1, \cdots, \vec{R}_N, t) \qquad (4.3.1)$$

Here m_n is the mass of the n^{th} sphere and $\vec{F}_n$ the force on the n^{th} sphere. After a very short time the distribution over the velocities becomes Maxwellian. The distribution in space takes much longer to relax to equilibrium. We may thus write after this short time period

$$P(\vec{w}_1, \cdots, \vec{w}_N, \vec{R}_1, \cdots, \vec{R}_N, t) = P(\vec{R}_1, \cdots, \vec{R}_N, t) \prod_{n=1}^{N} \left(\frac{m_n}{2\pi kT} \right)^{3/2} \exp\left(- \frac{m_n |\vec{w}_n|^2}{2kT} \right) \qquad (4.3.2)$$

Substitution in eq. (4.3.1) gives

$$\frac{\partial}{\partial t} P(\vec{R}_1, \cdots, \vec{R}_N, t) = \sum_{n=1}^{N} \vec{w}_n \cdot \left(\frac{1}{kT} \vec{F}_n - \frac{\partial}{\partial \vec{R}_n} \right) P(\vec{R}_1, \cdots, \vec{R}_N, t) \qquad (4.3.3)$$

When the distribution finally reaches equilibrium one has

$$\vec{F}_n P_{eq} = kT \frac{\partial}{\partial \vec{R}_n} P_{eq} \quad \Longrightarrow \quad \vec{F}_n = kT \frac{\partial}{\partial \vec{R}_n} \ln P_{eq} \qquad (4.3.4)$$

In equilibrium the force on the particle is cancelled by the so called <u>entropic force</u>. The velocity of the particle caused by this entropic force is according to eq. (4.2.

$$\vec{w}_n = - kT \sum_{m=1}^{n} \overleftrightarrow{\mu}_{nm} \cdot \frac{\partial}{\partial \vec{R}_n} \ln P_{eq} \qquad (4.3.5)$$

In general one may derive an equation of motion for

$$P(\vec{R}_1, \cdots, \vec{R}_N, t) \equiv \int d\vec{w}_1 \cdots d\vec{w}_N \; P(\vec{w}_1, \cdots, \vec{w}_N, \vec{R}_1, \cdots, \vec{R}_N, t) \qquad (4.3.6)$$

by integration of eq. (4.3.1) over the velocities. The first term on the right hand side gives zero as one may easily show by partial integration. The second term will contain some average velocity

$$\vec{w}_n^{\,av} = \int d\vec{w}_1 \cdots d\vec{w}_N \; \vec{w}_n \, P(\vec{w}_1, \cdots, \vec{w}_N, \vec{R}_1, \cdots, \vec{R}_N, t) \Big/ P(\vec{R}_1, \cdots, \vec{R}_N, t) \qquad (4.3.7)$$

The basic assumption is now that this velocity is simply the velocity due to the entropic force. Thus

$$\vec{w}_n^{\,av} = -kT \sum_{m=1}^{n} \vec{\vec{\mu}}_{nm} \cdot \frac{\partial}{\partial \vec{R}_n} \ln P \qquad (4.3.8)$$

One then obtains after integrating eq. (4.3.1) over the velocities

$$\frac{\partial}{\partial t} P(\vec{R}_1, \dots, \vec{R}_N, t) = kT \sum_{m,n=1}^{N} \frac{\partial}{\partial \vec{R}_n} \cdot \vec{\vec{\mu}}_{nm} \cdot \frac{\partial}{\partial \vec{R}_m} P(\vec{R}_1, \dots, \vec{R}_N, t) \qquad (4.3.9)$$

This is the <u>Smoluchowski equation</u> for the diffusion of N spheres in a fluid. The <u>diffusion coefficients</u> are

$$\vec{\vec{D}}_{nm} = kT \vec{\vec{\mu}}_{nm} = \frac{kT}{8\pi\eta R_{nm}} \left(\vec{\vec{I}} + \hat{R}_{nm} \hat{R}_{nm} \right) \quad \text{for } n \neq m \qquad (4.3.10)$$

while the <u>selfdiffusion coefficients</u> are given by

$$\vec{\vec{D}}_{nn} = kT \vec{\vec{\mu}}_{nn} = \frac{kT}{6\pi\eta a_n} \vec{\vec{I}} \quad \text{(Stokes - Einstein)} \qquad (4.3.11)$$

While the diffusion coefficients in general depend on the relative location of the particles, this is clearly not so for the selfdiffusion coefficient.

For a low density of spheres the regular diffusion coefficient, which describes the behaviour of the density of identical spheres ($a_n = a$)

$$\frac{\partial}{\partial t} n(\vec{r}, t) = D \Delta n(\vec{r}, t) \quad , \qquad (4.3.12)$$

is simply given by the self diffusion coefficient. Thus

$$D = \frac{kT}{6\pi\eta a} \qquad (4.3.13)$$

For higher values of the density the relation becomes rather complicated. We shall simply give the form of the result

$$D = N^{-1} \left\langle \sum_{n,m=1}^{N} \mu_{nm} \right\rangle n \left. \frac{\partial \varphi}{\partial n} \right|_{p,T} \qquad (4.3.14)$$

where $\langle \dots \rangle$ denotes the average over the configurations and φ the chemical potential for the spheres.

Diffusion

De beweging van diffunderende deeltjes wordt beschreven door de Langevin vergelijkingen

$$m \frac{d\vec{w}_i}{dt} = -\sum_{m=1}^{N} \vec{\vec{\zeta}}_{im} \cdot \vec{w}_m + \vec{K}_i \qquad (1)$$

De eerste term aan de rechterkant is de gebruikelijke wrijvingskracht die we ook in de vorige paragraaf besproken hebben. De tweede term is een fluctueerende kracht ook wel de Brownse kracht genoemd. De eigenschappen van deze Brownse kracht zijn uitgebreid besproken in het college statistische thermodynamica 2. We zullen dat nu niet in detail doen. De relaxatie van het systeem naar evenwicht gaat in twee fasen. Eerst gaat de snelheidsverdeling naar evenwicht en vervolgens de verdeling over de ruimte. Dit tweede stadium is het diffusie stadium waarin we in dit hoofdstuk geinteresseerd zijn. In dit stadium zijn de krachten met elkaar in evenwicht en reduceert vgl. (1) tot

$$\sum_{m=1}^{N} \vec{\vec{\zeta}}_{im} \cdot \vec{w}_m = \vec{K}_i \qquad \Rightarrow \qquad \vec{w}_i = \sum_{m=1}^{N} \vec{\vec{\mu}}_{im} \cdot \vec{K}_m \qquad (2)$$

We zijn nu geinteresseerd in de tijdsontwikkeling van de verdelingsfunctie van de deeltjes over de posities. In termen van de snelheden die door vgl. (2) gegeven worden hebben we

$$\frac{\partial}{\partial t} P(\vec{R}_1, \ldots, \vec{R}_N, t) = -\sum_{n=1}^{N} \frac{\partial}{\partial \vec{R}_n} \cdot \vec{w}_n \, P(\vec{R}_1, \ldots, \vec{R}_N, t) \qquad (3)$$

De snelheden zijn in evenwicht met de lokale verdeling van de posities. In feite kan men laten zien dat dit equivalent is met de volgende uitdrukking voor de Brownse kracht in vgl. (2)

$$\vec{K}_m (\vec{R}_1, \ldots, \vec{R}_N, t) = kT \frac{\partial}{\partial \vec{R}_m} \ln P(\vec{R}_1, \ldots, \vec{R}_N, t) \qquad (4)$$

Substitutie van vgl. (2) en (4) in (3) geeft

$$\frac{\partial}{\partial t} P(\vec{R}_1, \cdots, \vec{R}_N, t) = -kT \sum_{n,m=1}^{N} \frac{\partial}{\partial \vec{R}_n} \cdot \vec{\vec{\mu}}_{nm} \cdot \frac{\partial}{\partial \vec{R}_m} P(\vec{R}_1, \cdots, \vec{R}_N, t) \qquad (5)$$

De Brownse kracht lijkt qua vorm op de thermodynamische kracht. Het verschil is dat in dat geval de evenwichts distributie gebruikt wordt.

Om nu de zelfdiffusie coefficient te berekenen moeten we een vergelijking afleiden voor

$$P_1(\vec{R}_1, t) \equiv \int d\vec{R}_2 \cdots d\vec{R}_N \; P(\vec{R}_1, \cdots, \vec{R}_N, t) \qquad (6)$$

Als we in vgl. (5) over $\vec{R}_2, \cdots, \vec{R}_N$ integreren vinden we

$$\frac{\partial}{\partial t} P_1(\vec{R}_1, t) = F - kT(N-1) \frac{\partial}{\partial \vec{R}_1} \cdot \int d\vec{R}_2 \, \vec{\vec{\mu}}_{12} \cdot \frac{\partial}{\partial \vec{R}_2} P_2(\vec{R}_1, \vec{R}_2, t) \qquad (7)$$

waarbij P_2 gedefinieerd is door over $\vec{R}_3, \cdots, \vec{R}_N$ te integreren.
Nu definieeren we een conditionele waarschijnlijkheid door

$$P_2(\vec{R}_1, \vec{R}_2, t) \equiv g_2(\vec{R}_1, \vec{R}_2, t) \; P_1(\vec{R}_1, t) \qquad (8)$$

De aanname is nu dat er geen correlaties zijn zodat

$$g_2(\vec{R}_1, \vec{R}_2, t) = \frac{1}{V} \, \theta(R_{12} - 2a) \qquad (9)$$

Deze uitdrukking laat wel zien dat de deeltjes niet kunnen overlappen en verder dat de integraal een moet opleveren (het volume van een deeltje kan worden verwaarloosd in vergelijking met V)

$F \quad -\frac{kT}{6\pi\eta a} \Delta_{\vec{R}_1} P_1(\vec{R}_1, t)$

4.4 Effective viscosity

If one measures the viscosity of a suspension one finds a value which is in general appreciably higher than the value in the pure fluid. To understand that we consider eq. (3.1.5) for the flow

$$\frac{\partial}{\partial t}\vec{v} = -\text{grad}\,\frac{P}{\rho} - \frac{1}{\rho}\text{div}\,\overset{\Rightarrow}{\Pi} + \vec{G} \qquad (4.4.1)$$

In the flow experiment in which one measures the viscosity the flow is sufficiently slow to have a small Reynolds number. On the other hand it is sufficiently fast to make sedimentation velocities negligable. Thus $\vec{G}$ is a sum over the $\vec{G}_m$'s for separate spheres and the Stokes let strength can be neglected for each sphere. We thus have, neglecting all multipoles beyond the dipoles,

$$\vec{G}(\vec{r},t) = -8\pi\nu \sum_{m=1}^{N} \overset{\Rightarrow}{\alpha}_{D,m}(t)\cdot\vec{\nabla}\,\delta(\vec{r}-\vec{R}_m(t))$$

$$= -\vec{\nabla}\cdot\left[\;8\pi\nu\sum_{m=1}^{N}\overset{\Rightarrow}{\alpha}{}^{T}_{D,m}(t)\,\delta(\vec{r}-\vec{R}_m(t))\right] \qquad (4.4.2)$$

The T indicates the transpose of the matrix. Combining the viscous pressure term with the $\vec{G}$ we may define an effective viscous pressure by

$$\overset{\Rightarrow}{\Pi}_{eff}(\vec{r},t) = \overset{\Rightarrow}{\Pi}(\vec{r},t) + 8\pi\eta\sum_{m=1}^{N}\overset{\Rightarrow}{\alpha}{}^{T}_{D,m}\;\delta(\vec{r}-\vec{R}_m(t)) \qquad (4.4.3)$$

The Stokes dipoles may thus be absorbed in the pressure tensor and as a consequence it is clear that if one averages over the locations of the spheres one obtains a modified viscosity which expresses $\overset{\Rightarrow}{\Pi}_{eff}$ in terms of the velocity field.

We shall proceed to calculate this modification for small values of the volume fraction of the spheres. The radii are chosen equal. For small values of the volume fraction ϕ one has

$$\eta_{eff} = \eta\,(1 + c\,\phi) \qquad (4.4.4)$$

It is clear that the first term on the right hand side gives the zeroth order contribution. The second term determines the value of c. In order to calculate c explicitly we consider an incident velocity field of the form

$$\vec{v}_0(\vec{r}) = \vec{\vec{\alpha}}_0 \cdot \vec{r} \qquad \text{with } \vec{\vec{\alpha}}_0 \text{ symmetric traceless} \qquad (4.4.5)$$

a typical pure straining velocity field. Since we are only interested in the linear contribution it is sufficient to calculate $\vec{\vec{\alpha}}_D$ for one sphere placed in this velocity field. Take the center of the coordinate frame in the center of the sphere. The velocity field is given by

$$\vec{v}(\vec{r}) = \vec{\vec{\alpha}}_0 \cdot \vec{r} + \frac{1}{8\pi\nu} \int d\vec{r}\,' \, \vec{G}(\vec{r}\,') \cdot \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(1 - \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \right) \right] \qquad (4.4.6)$$

We now calculate the following average over the surface of the sphere

$$0 = \overline{\vec{v}(\vec{r})\,\vec{r}}^{\,S} = \frac{1}{3} a^2 \vec{\vec{\alpha}}_0 + \frac{1}{8\pi\nu} \int d\vec{r}\,' \, \vec{G}(\vec{r}\,') \cdot \overline{\left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(1 - \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \right) \vec{r} \right]}^{\,S} \qquad (4.4.7)$$

The average is straight forward to calculate. See paragraph (3.5) for a similar calculation. We will simply give the result

$$\overline{\left[\frac{1}{|\vec{r}-\vec{r}\,'|} \left(1 - \frac{(\vec{r}-\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \right) \vec{r} \right]}^{\,S} = + \frac{2}{5a} \vec{\vec{1}}\, \vec{r}\,' \quad \text{for } r' \leq a \qquad (4.4.8)$$

Substituting this into eq. (4.4.7) one obtains

$$\vec{\vec{\alpha}}_D \equiv \frac{1}{8\pi\nu} \int d\vec{r}\,' \, \vec{G}(\vec{r}\,')\, \vec{r}\,' = -\frac{5}{6} a^3 \, \vec{\vec{\alpha}}_0 \qquad (4.4.9)$$

The resulting effective pressure tensor is

$$\vec{\vec{\Pi}}_{eff} = -2\eta\, \vec{\vec{\alpha}}_0 - 5\eta\, \vec{\vec{\alpha}}_0 \left(\frac{4\pi}{3} a^3 \right) \sum_{m=1}^{N} \delta(\vec{r} - \vec{R}_m(t)) \qquad (4.4.10)$$

Averaging over the distribution of the spheres this reduces to

$$\vec{\vec{\Pi}}_{eff} = -2\eta \left(1 + \frac{5}{2}\phi \right) \vec{\vec{\alpha}}_0 \qquad (4.4.11)$$

We may thus identify as effective viscosity

$$\eta_{eff} = \eta \left(1 + \frac{5}{2}\phi\right) \qquad \underline{\text{Einstein}} \qquad (4.4.12)$$

The flow averaged over the distribution of the spheres satisfies

$$\frac{\partial}{\partial t}\vec{v} + \frac{1}{\rho}\,\mathrm{div}\,\vec{\vec{\Pi}}_{eff} = \frac{\partial}{\partial t}\vec{v} - \nu_{eff}\,\Delta\vec{v} = -\mathrm{grad}\,p \qquad (4.4.13)$$

For larger values of the volume fraction one must take the hydrodynamic interactions between the spheres into account. This is a rather difficult problem. It is however extremely important and in practice one finds that the effective viscosity diverges. In fact

$$\eta_{eff} = \frac{\eta}{1 - \frac{5}{2}\phi} \qquad (4.4.14)$$

gives a rather good qualitative description of the experimental result. One would obtain this formula if one calculates $\vec{\alpha}_D$ in the effective medium rather then in the original fluid. This procedure is not unlike mean field methods in various other fields. It should be realised, however, that such an analysis can only be expected to give a reasonable qualitative description. One should for instance not expect that eq. (4.4.4) predicts the location of the singularity with great accuracy.

4.5 Convective contributions to the force density

In most of the analysis up to now we have restricted ourselves to low Reynolds number flows. In fact the force density was defined as

$$\vec{\mathcal{G}} = \vec{F} - \vec{v}.\mathrm{grad}\,\vec{v} \qquad (4.5.1)$$

In low Reynolds numbers one only considers the $\vec{F}$ but not the $\vec{v}\cdot\text{grad}\,\vec{v}$ term. The reason to do this was two fold: on the one hand one has many situations for which low Reynolds number flow is sufficient and on the otherhand, as for instance in suspensions, makes the hydrodynamic interaction it allready difficult to determine $\vec{F}$ in a sufficiently accurate way. If one would try to account for the $\vec{v}\cdot\text{grad}\,\vec{v}$ term at the same time it would become a problem of such complexity that a general discussion becomes impractical.

As we have discussed the construction of $\vec{F}$ neglecting $\vec{v}\cdot\text{grad}\,\vec{v}$ for some cases we shall now consider the other extreme where we use $\vec{F}$ as a given quantity and discuss how to take the $\vec{v}\cdot\text{grad}\,\vec{v}$ into account. For this purpose we use the general solution, cf. eq. (3.9.11)

$$\vec{v} = \frac{1}{8\pi\nu}\,\vec{\vec{T}}\cdot\vec{g} \qquad (4.5.2)$$

To make the analysis algebraïcally simpler the dependence on $\vec{k}, \omega$ has not been indicated explicitly. Interpreting $\vec{\vec{T}}$ as an operator which gives the velocity field in terms of the source one may in fact also write eq. (3.9.12) in the form given in the above equation. $\vec{\vec{T}}$ is simply a socalled diagonal operator as a function of $\vec{k}$ and a convolution operator as a function of $\vec{r}$. Substituting eq. (4.5.1) into eq. (4.5.2) one has

$$\vec{v} = \frac{1}{8\pi\nu}\,\vec{\vec{T}}\cdot\left(\vec{F} - \vec{v}\cdot\text{grad}\,\vec{v}\right) \qquad (4.5.3)$$

The methods to analyse such an equation are known from field theory. The practical application of such methods has been most popular in high energy physics. In recent years it has been realised that the usefullness of such methods is not restricted to high energy physics but may also be used to analyse the above equation. The basic method to solve the above equation is by iteration. As a first step we define

$$\vec{V}_0 = \frac{1}{8\pi\nu} \overset{\Rightarrow}{T} \cdot \vec{F} \qquad (4.5.3)$$

This solution of the linear equation we then use as the starting point of an iteration procedure. The first step in this procedure is

$$\vec{V}_1 = \vec{V}_0 - \frac{1}{8\pi\nu} \overset{\Rightarrow}{T} \cdot (\vec{V}_0 \cdot \operatorname{grad} \vec{V}_0) \qquad (4.5.4)$$

The second step is

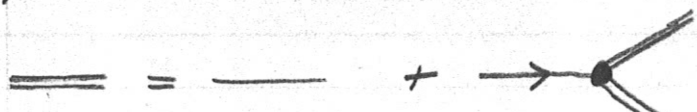
$$\vec{V}_2 = \vec{V}_0 - \frac{1}{8\pi\nu} \overset{\Rightarrow}{T} \cdot (\vec{V}_1 \cdot \operatorname{grad} \vec{V}_1) \qquad (4.5.5)$$

The i^{th} step is

$$\vec{V}_i = \vec{V}_0 - \frac{1}{8\pi\nu} \overset{\Rightarrow}{T} \cdot (\vec{V}_{i-1} \cdot \operatorname{grad} \vec{V}_{i-1}) \qquad (4.5.6)$$

It is clear that $\lim_{i\to\infty} \vec{V}_i$ will give the exact solution of the original equation. In practical situations it is sufficient to iterate only a few times as the lowest order contributions allready give a good idea of the often striking changes in the behaviour.

In order to understand and manipulate the iteration in a simple way one uses diagrams. In diagramatic form one writes eq. (4.5.3) as

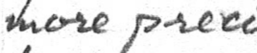


(4.5.7)

here $\vec{V}$ = ═══ , $\vec{V}_0$ = ——— , $\overset{\Rightarrow}{T}$ = →

and • = $-\frac{1}{8\pi\nu}$ grad

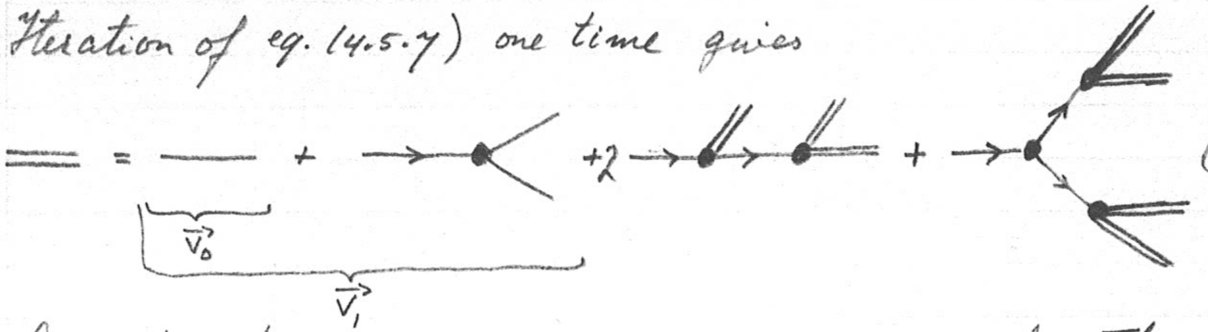
Of course the • should actually be defined somewhat more precise. As we shall not really do any calculations we will not have any need for a precise definition. Equation (4.5.3) can be written in diagramatic language as



(4.5.8)

Here $\frac{1}{8\pi\nu} \vec{F}$ = ×

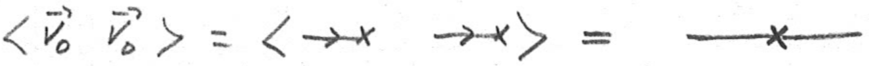
Iteration of eq. (4.5.7) one time gives



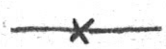
(4.5.9)

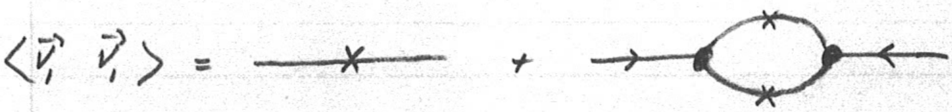
Generating higher order diagrams is very simple. These diagrams are similar to Feynmann diagrams. A more precise definition of the lines and the dots is analogous to the Feynmann rules.

One of the aspects which is conveniently analysed in terms of these diagrams are thermal fluctuations. In that case the force $\vec{F}$ is the source of these fluctuations. If one calculates the autocorrelation function of $\vec{v}_0$ one finds the simple exponential decay of the velocity autocorrelation function usually used



(4.5.10

Here  is again a shorthand notation for the zeroth order velocity autocorrelation function. If one goes to the next order one obtains



(4.5.11)

where we note that the average of three crosses is zero. The new contribution decays as $t^{-3/2}$ for long times and we thus see that for sufficiently large times the correlatio function is dominated by this second contribution rather then the first. This remarkable property, which can be understood so easily in the context of hydrodynamic was in fact discovered about 15 years ago in molecular dynamical calculations for liquids In such calculation one solves Newton's equations for about 500 - 1000 particle (hard spheres usually) in a box. The amazing aspect is tha even on the molecular level the longtime behaviour is dominated by the hydrodynamic (long wave length) modes.